198:538 Complexity of Computation
Rutgers University, Spring 2007

Homework 2
due Tuesday 20 February

Problem 1

In this problem you prove an analog of the Karp-Lipton-Sipser theorem for counting problems.

We say that a circuit C solves #3SAT on n variables if for every 3CNF φ on n variables, $C(\varphi)$ outputs the number of satisfying assignments for φ. We say #3SAT has a polynomial-size family of circuits if there is a polynomial p such that for every n, there exists a circuit of size $p(n)$ that solves #3SAT on n variables.

(a) Show that the following language is in coNP:

$$L = \{(C, 1^n) : C \text{ solves \#3SAT on } n \text{ variables}\}$$

(b) Show that if #3SAT has a polynomial-size family of circuits, then $\mathrm{P}^{\#\mathrm{P}} = \Sigma_2$.

Problem 2

Recall that the number of witnesses of an NP relation can be approximately counted by a randomized procedure with access to a SAT oracle. Show that unless the polynomial hierarchy collapses, exact counting is harder than approximate counting: If $\mathrm{P}^{\#\mathrm{SAT}} \subseteq \mathrm{BPP}^{\mathrm{SAT}}$ then $\Sigma_3 = \Pi_3$.

Problem 3

Show that there exists a randomized algorithm that given access to a SAT oracle, runs in expected polynomial time and on input a satisfiable CNF φ, outputs a *uniformly random* satisfying assignment for φ. (Namely, if φ has s satisfying assignments $a_1, \ldots, a_s$, then the algorithm outputs a_i with probability exactly $1/s$.)

Problem 4

A boolean formula φ is in DNF, or disjunctive normal form, if it is written as a disjunction of conjunctions of literals, for instance

$$(x_1 \wedge \overline{x}_3) \vee x_2 \vee (\overline{x_1} \wedge x_2 \wedge x_4).$$

Let #DNF be the problem of counting the number of satisfying assignments of a DNF φ.

(a) Show that #DNF is #P-complete.

(b) Show that there exists a randomized algorithm that runs in expected polynomial time and on input a DNF φ, uniformly samples a satisfying assignment for φ. (Therefore #DNF can be approximated by a randomized algorithm, no oracles involved.)

Hint: Write $\varphi = c_1 \vee \cdots \vee c_m$. Let Ω be the set $\{0,1\}^n \times \{1, \ldots, m\}$, where n is the number of variables in φ. Consider the following two subsets of Ω:

$$A = \{(a, i) : \text{Assignment } a \text{ satisfies clause } c_i\}$$
$$B = \{(a, i) : \text{Assignment } a \text{ satisfies clause } c_i \text{ but none of } c_1, \ldots, c_{i-1}\}.$$

Show how to sample uniformly from A, and use the fact that a substantial fraction of samples in A also fall inside B.