CSI 2101: Discrete Structures — Final Examination Solutions

University of Ottawa, Winter 2024

1. Are the following propositions true or false? If a proposition is true, prove it. If it is false, prove its negation. Specify your proof method. m and n are integers.

   (a) For every m there exists a n such that $m + 2n = mn$.

   Solution: False. We show a counterexample. If $m = 5$ then n must satisfy $5 + 2n = 5n$, so n must equal $3/5$ which is not an integer.

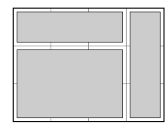
   (b) There exists an m such that for every n, $m + 2n = mn$.

   Solution: False. For contradiction suppose such an m existed. Then $m(1 - n) + 2n = 0$ for all n. But when $n = 1$, the left hand side equals 2 and the right hand side equals zero so $2 = 0$, a contradiction.

2. Prove that $2n + 3 \leq 2^n$ for all integer $n \geq 4$ using induction.

   Solution: In the base case $n = 4$, $2 \cdot 4 + 3 = 11 \leq 16 = 2^4$. For the inductive step assume $2n + 3 \leq 2^n$ for a given $n \geq 3$. Then $2(n + 1) + 3 = 2n + 3 + 2 \leq 2^n + 2 \leq 2^n + 2^n = 2^{n+1}$.

3. Let $f(n)$ be the number of ways to tile a $3 \times n$ field using 1×3 and 2×3 tiles. An example tiling for $n = 4$ is shown on the right.



   (a) Fill in the blanks:

   Solution:

   $$f(0) = \underline{1} \qquad f(1) = \underline{1} \qquad f(2) = \underline{2}$$

   When $n = 0$, the empty tiling is the only possible tiling. When $n = 1$, there is one tiling with a vertical 3×1 tile. When $n = 2$, there are two tilings: One with two vertical 3×1 tiles and one with a single 3×2 tile.

   (b) Write a recurrence for $f(n)$ in terms of $f(n - 1)$, $f(n - 2)$, and $f(n - 3)$. Explain your answer.

   Solution: $f(n) = f(n - 1) + f(n - 2) + 3f(n - 3)$. The set of tilings is a disjoint union of those that start with a 3×1 vertical tile, those that start with a 3×2 vertical tiles, and those that fill up the first three columns with horizontal tiles. There are $f(n - 1)$, $f(n - 2)$, and $3f(n - 3)$ of each type, respectively. By the sum rule we obtain the above recurrence.

   (c) Calculate $f(5)$.

   Solution: We iterate the recurrence to obtain $f(3) = 2 + 1 + 3 \cdot 1 = 6$, $f(4) = 6 + 2 + 3 \cdot 1 = 11$, and $f(5) = 11 + 6 + 3 \cdot 2 = 23$.

4. In how many ways can you place 10 white balls and 10 black balls in a 2×10 grid so that there are

   (a) equally many white and black balls in every row?

   Solution: Each row must contain five white balls and five black balls. There are $\binom{10}{5}$ ways to arrange the balls in the first row (the arrangement can be viewed as a string in $\{W, B\}^{10}$ with five Ws). There are as many in the second row. By the product rule the total number is $\binom{10}{5}^2$.

   (b) equally many white and black balls in every column?

   Solution: There are 2^{10} possible arrangements of the first row (the arrangement can be any string in $\{W, B\}^{10}$). Once the first row is fixed, there is exactly one possible arrangement of the second row obtained by swapping the color in each column. By the generalized product rule the total number is 2^{10}.

(c) equally many white and black balls in every row and in every column?

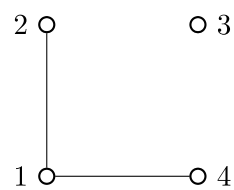
Solution: There are not $\binom{10}{5}$ possible arrangements of the first row as there must be exactly five white balls. Once the first row is fixed, there is again exactly one possible arrangement of the second row. By the generalized product rule the number is $\binom{10}{5}$.

Justify your answer using suitable counting rules.

5. The vertices of G_n are the numbers $1, 2, \ldots, n$. The edges are pairs $\{x, y\}$ for which $x^3 + y^3 \equiv 1 \pmod 4$.

(a) Draw a diagram of G_4.

Solution: As $1^3 \equiv 1$, $3^3 \equiv (-1)^3 \equiv -1$, and $2^3 \equiv 4^3 \equiv 0$ modulo 4, the edges are $\{1, 2\}$ and $\{1, 4\}$.



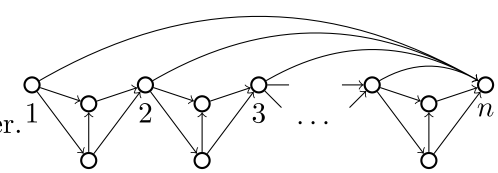
(b) Show that G_n is not connected for all $n \geq 4$. (**Hint:** Which partition A, B has no edges from A to B?)

Solution: One such partition is $A = \{3\}$ and $B = \{1, \ldots, n\} \setminus \{3\}$. As the numbers in (a) cover all the equivalence classes modulo 4 it must be that $b^3 \equiv 0$, 1, or 3 for every $b \in B$, so $3^3 + b^3 \equiv 3$, 0, or 2. Therefore the vertex 3 is isolated so G_n is not connected.

(c) Show that G_n is bipartite for all n. (**Hint:** Which partition C, D has no edges within C and D?)

Solution: Let C be the even vertices and D be the odd vertices in $\{1, \ldots, n\}$. There is no edge within C because if c and c' are even then $c^3 + c'^3$ is also even so it cannot be 1 modulo 4. Similarly there is no edge within D because if d and d' are odd then $d^3 + d'^3$ is even so it cannot be 1 modulo 4 either.

6. Let $f(n)$ be the number of directed paths from 1 to n in the graph on the right.



(a) Fill the blanks in the following recurrence. Explain your answer.

Solution:

$$f(n) = \underline{3} \cdot f(n-1) + \underline{1}.$$

The set of paths from 1 to n through vertex 2 is a product set specified by a path from 1 to 2 and a path from 2 to n. There are three paths from 1 to 2 and $f(n-1)$ paths from 2 to n. By the product rule there are $3f(n-1)$ such paths. Accounting for the additional direct edge from 1 to n gives the above formula (by the sum rule).

(b) Solve the recurrence from part (a).

Solution: We can homogenize the recurrence to $f(n) + a = 3(f(n-1) + a)$. As $3a - a$ must equal 1 we obtain $a = 1/2$. Unwinding the recurrence we get $f(n) + 1/2 = 3^{n-1}(f(1) + 1/2)$. When $n = 1$, the graph has a single vertex and $f(1) = 1$ as the empty path is the only path. Therefore $f(n) = 3^n/2 - 1/2$. Alternatively, unwinding the recurrence gives

$$f(n) = 3^{n-1} f(1) + 3^{n-2} + \cdots + 3 + 1 = 3^{n-1} + \frac{3^{n-1} - 1}{2} = \frac{3^n - 1}{2}.$$

BONUS. Let $f(n)$ be the number of ways to tile a $3 \times n$ field with 1×2 tiles. What is $f(n)$ mod 2?

Solution: Let $g(n)$ is the number of ways to tile a $3 \times n$ field with the top left square missing. Then

$$f(n) = f(n-2) + 2g(n-1)$$

because the set of all tilings can be written as a direct sum depending on the tile that occupies the leftmost middle slot. If this tile is horizontal so must be the tile above and below it and there are $f(n-2)$ ways to tile the rest of the field. If it is vertical the remaining slot in the leftmost column must be occupied by a horizontal tile and there are $g(n-1)$ ways to tile the rest of the field. As there are two possible covering of this slot by a vertical tile we obtain the above recurrence.

Taking both sides modulo two we obtain $f(n) \equiv f(n-2) \pmod 2$, so $f(n) \equiv f(0)$ if n is even and $f(1)$ if n is odd. As $f(0) = 1$ and $f(1) = 0$, $f(n)$ has the opposite parity of n, or

$$f(n) = n + 1 \bmod 2.$$