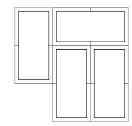
1. Prove the following using induction for every positive integer n.

   (a) The sum of the first n odd integers is n^2.

   (b) $1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2 \leq 2 - 1/n$.

   (c) $4^n/2n \leq (2n)!/(n!n!) \leq 4^n$.

   (d) If n is odd, an $n \times n$ grid with a corner square removed can be tiled using 2×1 blocks. (See example for $n = 3$. Your proof may describe a different tiling.)



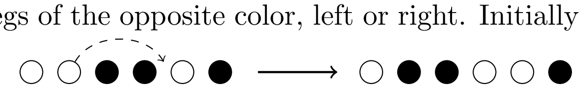
2. Use strong induction to prove the following for all positive integer n.

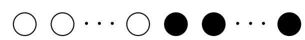
   (a) If $n \geq 14$ then n can be written as $4a + 5b$ for some integers $a, b \geq 0$.

   (b) $(3/2)^{n-2} \leq F_n \leq (7/4)^{n-2}$ for every $n \geq 4$, where F_n is the n-th Fibonacci number ($F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$.)

   (c) $G_n = n!$, where G_n is given by $G_1 = 1$, $G_{n+1} = 1 + G_1 + 2G_2 + \cdots + nG_n$. (**Hint:** $n \cdot n! = (n+1)! - n!$)

3. n white pegs and n black pegs are arranged in a line. In each step you are allowed to move any peg past *two* consecutive pegs of the opposite color, left or right. Initially all white pegs are to the left of the black ones.

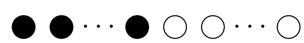




INITIAL

   (a) Assume n is odd. Say a pair of pegs is *inverted* if one is black, one is white, and the black one is to the left of the right one. Prove that "the number of inverted pairs is even" is an invariant.

   (b) If n is odd, can the colors be reversed so that all black pegs are to the left of all white ones?



   (c) If n is even, can the colors be reversed?

4. You have a system of n switches, each of which can be in one of two states: OFF or ON. There are 2^n possible configurations of this system. For example, when $n = 2$ the four possible configurations for the pair of switches are (OFF, OFF), (OFF, ON), (ON, OFF), and (ON, ON).

   Initially, all switches are OFF. In each step, you are allowed to flip exactly one of the switches. Is there a sequence of flips that makes each possible configuration arise exactly once? For example when $n = 2$ this sequence of flips has the desired property (the number on the arrow indicates the switch that is flipped):

$$(\text{OFF}, \text{OFF}) \xrightarrow{2} (\text{OFF}, \text{ON}) \xrightarrow{1} (\text{ON}, \text{ON}) \xrightarrow{2} (\text{ON}, \text{OFF})$$

   (a) Show a sequence of flips that works when $n = 3$.

   (b) Use induction to prove that for every $n \geq 1$, there exists a sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration. (**Hint:** You may need to strengthen the proposition.)

   (c) Now suppose that you flip not one but two switches at a time. Use an invariant to prove that for every $n \geq 2$, there is no sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration.