Practice Final 1

1. Prove that for every integer n there exists an integer k such that $|n^2 - 5k| \leq 1$. (**Hint:** What is $n^2 \bmod 5$?)

2. Alice places two pebbles at the opposite corners of an 8 by 8 chessboard. At each step, she can

   - put a new pebble in an empty square, if *exactly one* of its neighbors contains a pebble, or
   - remove a pebble from a square, if *at least one* of its neighbors contains a pebble.

   Neighbors are squares that share a common side.

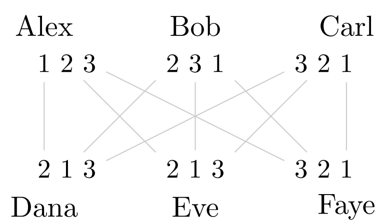
   (a) Define a suitable graph G for which "G has two or more connected components" is an invariant. Prove the invariant.

   (b) Can the board ever have a single pebble on it?

3. Sort these three functions in increasing order of growth: $\sqrt{n} \cdot \log n$, $n/\sqrt{\log n}$, $\sqrt{n \cdot \log n}$. For your sorted list f, g, h show that f is $o(g)$ and g is $o(h)$.

4. What is the multiplicative inverse of 100 modulo 1009? Show your work.

5. Find a stable matching for these preferences and show that there is no other stable matching.



6. The number of length-n strings with symbols $\{\texttt{A}, \texttt{B}, \texttt{C}\}$ in which no symbol appears consecutively three times (i.e., the patterns `AAA`, `BBB`, `CCC` are forbidden) is $\Theta(a^n)$.

   (a) Write a recurrence for the number $f(n)$ of such strings that start with a fixed symbol (say an `A`).

   (b) Find the number a.

Practice Final 2

1. What is $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + 1000)$?

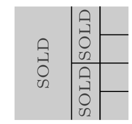
2. A department has 10 men and 15 women. How many ways are there to form a committee with six members if it must have...

   (a) the same number of men and women?

   (b) at least one man and at least one woman?

3. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

4. An $n \times n$ plot of land (n is a power of two) is split in two equal parts by a North-South fence. The Western half is sold and the Eastern half is split in two equal parts by an West-East fence. The same procedure is applied to the remaining $(n/2) \times (n/2)$ plots until 1×1 plots are obtained (see $n = 4$ example).



   (a) Let $T(n)$ be the units of fence used. Write a recurrence for $T(n)$.

   (b) Solve the recurrence.

   (c) Prove that your answer is correct using induction.

5. Let G be the following graph. The vertices of G are all the integers between -10 and 10 except for 0 (20 vertices in total). The pair $\{x, y\}$ is an edge of G if (and only if) $-30 < xy < 0$.

   (a) Show that G is bipartite.

   (b) Show that G does not have a perfect matching.

6. A *cut-edge* in a connected graph is an edge e such that if e was removed, the graph would no longer be connected. Show that any connected graph in which all vertices have even degree does not have a cut-edge.

Practice Final 3

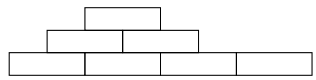
1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".

2. The sequence $f(n)$ is given by $f(n+1) = 2^{f(n)}$ for $n \geq 1$ with $f(0) = 2$.

   (a) Calculate $f(n)$ mod 5 for $n = 1$, $n = 2$, and $n = 3$.

   (b) Give a formula for $f(n)$ mod 5 for all $n \geq 4$. Justify your answer.

3. Blocks of height one are stacked in layers in some formation. Each layer has strictly fewer blocks than the one under it. For example the 7-block formation below has height 3. Show that the height of an n-block formation is $O(\sqrt{n})$.



4. You drop 30 balls into 7 urns. Some of the balls are red and some are blue. Show that at least three balls of the same colour land in the same urn.

5. G is a directed graph whose vertices are the integers from -10 to 10 (inclusive) and whose edges (x, y) are those ordered pairs for which $|x| - |y| = 1$. For each of the following claims, say if it is true or false and provide a proof.

   (a) G has a path of length 10.

   (b) G has a parallel schedule of duration 11.

   (c) G has an antichain of size 6.

6. The vertices of graph H_n are the n integers from $-n$ to n except 0. The edges of H_n are the pairs $\{x, y\}$ such that $x = -y$ or $|y - x| = 1$.

   (a) Show that H_n is bipartite.

   (b) How many perfect matchings do H_1 and H_2 have?

   (c) How many perfect matchings does H_{10} have? (**Hint:** Write a recurrence.)

Practice Final 4

1. Let $P(n)$ be the statement "There exists an $n \times n$ table of numbers in which the sum of every row is even and the sum of every column is odd".

   (a) Prove that $P(2)$ is true. Specify your proof method.

   (b) Prove that $P(3)$ is false. Specify your proof method.

2. Let $B_0 = 0$, $B_1 = 1$, and $B_n = B_{n-1} + \frac{1}{2}B_{n-2}$ for all $n \geq 2$. Prove that $B_n \geq ((1+\sqrt{3})/2)^{n-2}$ for all $n \geq 1$. Specify your proof method.

3. Find integers x and y between 0 and 16 that satisfy the following equations. Justify your steps.

$$4x + 3y \equiv 2 \pmod{17}$$
$$3x + 4y \equiv 3 \pmod{17}.$$

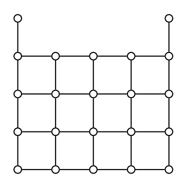
4. Does there exist a graph G with five vertices a, b, c, d, e such that

   (a) The degrees of the five vertices are

v	a	b	c	d	e
deg v	3	3	2	1	1

   (b) The degrees are as in part (a) and G is bipartite?

5. Let G be the following graph.



   (a) Show that G is bipartite.

   (b) Does G have a perfect matching? Justify your answer.

6. Alice, Bob, and Charlie play a game. Initially Alice holds \$1, Bob holds \$2, and Charlie holds \$5. In each round every player splits their holdings evenly in two and gives them away to the other two players.

   (a) Let $a(n)$ be Alice's holdings after n rounds. Calculate $a(1)$ and $a(2)$.

   (b) Show that $a(n+1) = 4 - a(n)/2$. (**Hint:** The sum of all players' holdings remains invariant.)

   (c) Solve the recurrence from part (b) with initial condition $a(0) = 1$ by unfolding or homogenization.