Practice Final 1

1. Prove that for every integer n there exists an integer k such that $|n^2 - 5k| \le 1$. (**Hint:** What is $n^2 \bmod 5$?)

   Solution: First we check that for all n, $n^2 \bmod 5$ equals 0, 1 or 4:

$n \bmod 5$	0	1	2	3	4
$n^2 \bmod 5$	0	1	4	4	1

   Since $4 \equiv -1 \pmod 5$ it follows that for every n, n^2 is congruent to 0, 1, or -1 modulo 5. Therefore n^2 is of the form $5k$ or $5k-1$ or $5k+1$ for some integer k. In all cases $|n^2 - 5k| \le 1$.

2. Alice places two pebbles at the opposite corners of an 8 by 8 chessboard. At each step, she can

   - put a new pebble in an empty square, if *exactly one* of its neighbors contains a pebble, or
   - remove a pebble from a square, if *at least one* of its neighbors contains a pebble.

   Neighbors are squares that share a common side. Can the board ever have a single pebble on it?

   (a) Define a suitable graph G for which "G has two or more connected components" is an invariant. Prove the invariant.

   Solution: G is the graph whose vertices are the pebbles and edges are pebbles on neighboring squares. Initially G has two connected components so the invariant holds. We show that the number of connected components cannot decrease after a transition. For the first type of move it remains the same because the new pebble extends an existing component without affecting the others. For the second type of move, the component that the removed pebble belongs to may break up up into one or more components, while the other components are unaffected, so the number of components cannot decrease.

   (b) Can the board ever have a single pebble on it?

   Solution: No, because the corresponding graph has one vertex and therefore only one connected component.

3. Sort these three functions in increasing order of growth: $\sqrt{n} \cdot \log n$, $n/\sqrt{\log n}$, $\sqrt{n \cdot \log n}$. For your sorted list f, g, h show that f is $o(g)$ and g is $o(h)$.

   Solution: $\sqrt{n \log n}$ is $o(\sqrt{n} \log n)$ because the ratio $\sqrt{n \log n}/\sqrt{n} \log n$ equals $1/\sqrt{\log n}$, which eventually becomes and stays smaller than any given constant. $\sqrt{n} \log n$ is $o(n/\sqrt{\log n})$ because the ratio $\sqrt{n} \log n/(n/\sqrt{\log n})$ equals $(\log n)^{3/2}/n^{1/2}$. In Lecture 7 we showed that $(\log n)^a$ is $o(n^b)$ for any constants $a, b > 0$, so this ratio becomes and stays smaller than any constant when n is sufficiently large.

4. What is the multiplicative inverse of 100 modulo 1009? Show your work.

   Solution: We look for numbers s and t such that $100s + 1009t = 1$. The extended Euclid's algorithm goes through the steps

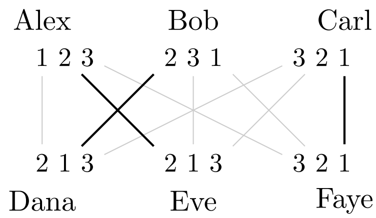
$$1009 = 10 \cdot 100 + 9$$
$$100 = 11 \cdot 9 + 1,$$

   from where

$$1 = 100 - 11 \cdot 9 = 100 - 11 \cdot (1009 - 10 \cdot 100) = 111 \cdot 100 - 11 \cdot 1009$$

   so we can set $s = 111$ and $t = -11$. Therefore $100 \cdot 111 \equiv 1 \mod 1009$ and 111 is the desired multiplicative inverse.

5. Find a stable matching for these preferences and show that there is no other stable matching.



Solution: Consider the marked matching $\{\text{Alex}, \text{Eve}\}, \{\text{Bob}, \text{Dana}\}, \{\text{Carl}, \text{Faye}\}$. We show that no other matching is stable. As a stable matching always exists, this one must be stable.

In any stable matching, Carl must be matched to Faye because they are each other's first choice (so they would be a rogue couple if not matched). For the rest, the matching $\{\text{Alex}, \text{Dana}\}, \{\text{Bob}, \text{Eve}\}$ can be ruled out because Bob and Dana would be a rogue couple. This leaves the above matching as the only stable possibility.

Alternative solution: If we run the Gale-Shapley algorithm, on day 1 Alex proposes to Dana and Bob and Carl propose to Faye. Faye picks Carl, so on day 2 both Alex and Bob propose to Dana. Dana picks Bob, so the final matching is $\{\text{Alex}, \text{Eve}\}, \{\text{Bob}, \text{Dana}\}, \{\text{Carl}, \text{Faye}\}$. We proved in Lecture 5 that this is stable.

Let us now run the Gale-Shapley algorithm again, but with the girls doing the proposing this time around. On day 1 Dana and Eve propose to Bob and Faye proposes to Carl. Carl picks Faye and Bob picks Dana over Eve. On day 2 Eve proposes to Alex resulting in the same final stable matching.

By Theorem 6 in Lecture 8, the first matching is the best possible for the boys (every boy gets his best possible choice among all stable matchings), while the second one is the worst possible for the boys (every boy gets his worst possible choice). Since they are the same there can be only one stable matching.

6. The number of length-n strings with symbols $\{\mathtt{A}, \mathtt{B}, \mathtt{C}\}$ in which no symbol appears consecutively three times (i.e., the patterns `AAA`, `BBB`, `CCC` are forbidden) is $\Theta(a^n)$.

   (a) Write a recurrence for the number $f(n)$ of such strings that start with a fixed symbol (say an `A`).

   Solution: If the second symbol is *not* an `A` then the remaining part can be any string of the same type that starts with a `B` or a `C`, so there are $2f(n-1)$ choices for it. If the second symbol is an `A` then the third symbol must be a `B` or a `C` and there are $2f(n-2)$ choices for the remaining part. Therefore f satisfies the recurrence $f(n) = 2f(n-1) + 2f(n-2)$.

   (b) Find the number a.

   Solution: Removing the restriction that the string starts with a fixed symbol triples the count so it has the same big-theta asymptotics as $f(n)$. The recurrence solution $f(n)$ is a linear combination of x_1^n and x_2^n, where x_1 and x_2 are the roots of $x^2 = 2x + 2$. The two roots are $x_1 = 1 + \sqrt{3}$ and $x_2 = 1 - \sqrt{3}$. As x_2 is less than 1 it must be that $f(n)$ is $\Theta(x_1)^n$ so $a = 1 + \sqrt{3}$.

Practice Final 2

1. What is $1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+1000)$?

Solution: The sum of the first k integers is $k(k+1)/2 = \frac{1}{2}k^2 + \frac{1}{2}k$, so

$$\begin{aligned} 1 + (1+2) + \cdots + (1+2+3+\cdots+n) &= \tfrac{1}{2}(1^2 + 2^2 + \cdots + n^2) + \tfrac{1}{2}(1+2+\cdots+n) \\ &= \tfrac{1}{2}(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n) + \tfrac{1}{2}(\tfrac{1}{2}n^2 + \tfrac{1}{2}n) \\ &= \tfrac{1}{6}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{3}n \end{aligned}$$

using the formulas for the sum of the first n squares of integers and the sum of the first n integers, respectively. (Notice that this expression gives the correct answer for $n = 0$, 1, and 2.) Plugging in $n = 1000$ we obtain the answer $\frac{1}{6} \cdot 10^9 + \frac{1}{2} \cdot 10^6 + \frac{1}{3} \cdot 10^3 = 167,167,100$.

Alternative solution: as $1+2+\cdots+n = n(n+1)/2$ we may guess that $1+(1+2)+\cdots+(1+2+3+\cdots+n)$ has the form $an^3 + bn^2 + cn + d$. Plugging in $n = 0, 1, 2, 3$ we get that a, b, c, d must satisfy

$$\begin{aligned} d &= 0 \\ a + b + c + d &= 1 \\ 8a + 4b + 2c + d &= 1 + (1 + 2) = 4 \\ 27a + 9b + 3c + d &= 4 + (1 + 2 + 3) = 10 \end{aligned}$$

Eliminating first d and then c we get $6a+2b = 2$ and $24a+6b = 7$. This solves to $2b = 1$. Therefore $b = 1/2$. Plugging back in we get $a = 1/6$ and $c = 1/3$.

We now verify that the sum equals $\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$ by induction on n. The base case $n = 1$ was already checked. As for the inductive step we assume the claim is true for n and verify it for $n + 1$:

$$\begin{aligned} 1 + (1 + 2) + \cdots + (1 + \cdots + (n + 1)) &= \tfrac{1}{6}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{3}n + (1 + \cdots + (n + 1)) \\ &= \tfrac{1}{6}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{3}n + \frac{(n + 1)(n + 2)}{2} \\ &= \tfrac{1}{6}(n^3 + 3n^2 + 3n + 1) + \tfrac{1}{2}(n^2 + 2n + 1) + \tfrac{1}{3}(n + 1) \\ &= \tfrac{1}{6}(n + 1)^3 + \tfrac{1}{2}(n + 1)^2 + \tfrac{1}{3}(n + 1). \end{aligned}$$

Plugging in $n = 1000$ we obtain the same answer as above.

2. A department has 10 men and 15 women. How many ways are there to form a committee with six members if it must have...

   (a) the same number of men and women?

   Solution: We apply the product rule. There are $\binom{10}{3}$ ways to choose the three men in the committee and $\binom{15}{3}$ ways to choose the three women in it, so the number of possible committees is $\binom{10}{3} \cdot \binom{15}{3}$.

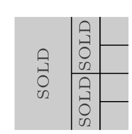
   (b) at least one man and at least one woman?

   Solution: The number of possible committees is $\binom{25}{6}$, the number of men-only committees is $\binom{10}{6}$, and the number of women-only committees is $\binom{15}{6}$. By the sum rule the number is $\binom{25}{6} - \binom{15}{6} - \binom{10}{6}$.

3. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

   Solution: We can prove this proposition by cases depending on the residue of n^3+n modulo 3. If $n \equiv 0 \bmod 3$ then $n^3 + n$ is divisible by 3, while $2n^3 + 1 \equiv 1 \bmod 3$, so $2n^3 + 1$ is not divisible by 3, so the proposition holds. If $n \equiv 1 \bmod 3$ then $n^3 + n \equiv 2 \bmod 3$, so $n^3 + n$ is not divisible by 3 and the proposition holds again. If $n \equiv 2 \bmod 3$, then $n^3 + n \equiv 2 \bmod 3$ and $n^3 + n$ is not divisible by 3 again.

4. An $n \times n$ plot of land (n is a power of two) is split in two equal parts by a North-South fence. The Western half is sold and the Eastern half is split in two equal parts by an West-East fence. The same procedure is applied to the remaining $(n/2) \times (n/2)$ plots until 1×1 plots are obtained (see $n = 4$ example). How many units of fence are used?



   (a) Let $T(n)$ be the units of fence used. Write a recurrence for $T(n)$.

   Solution: The amount of fence for an $n \times n$ field is twice the amount used for a field half the size plus n units of vertical fence plus $n/2$ units of horizontal fence, giving the recurrence $T(n) = 2T(n/2)+3n/2$ for $n > 1$. The initial condition is $T(1) = 0$.

(b) Solve the recurrence.

Solution: We can unwind the recurrence as follows:

$$\begin{aligned} T(n) &= 2T(n/2) + 3/2 \cdot n \\ &= 2(2T(n/2^2) + 3/2 \cdot n/2) + 3/2 \cdot n = 2^2T(n/2^2) + 3/2 \cdot 2n \\ &= 2^2(2T(n/2^3) + 3/2 \cdot n/2^2) + 3/2 \cdot 2n = 2^3T(n/2^3) + 3/2 \cdot 3n \end{aligned}$$

After $\log n$ steps we expect to obtain $T(n) = n \cdot T(1) + \frac{3}{2}n \log n = \frac{3}{2}n \log n$.

(c) Prove that your answer is correct using induction.

For the base case $n = 1$, $T(1) = 0$ as desired. For the inductive step we assume $T(k) = \frac{3}{2}k \log k$ for all $k < n$ that are powers of two. Then

$$T(n) = 2T(n/2) + 3n/2 = 2 \cdot \frac{3}{2} \cdot \frac{n}{2} \log(n/2) + \frac{3n}{2} = \frac{3n}{2} \cdot (\log n - 1) + \frac{3n}{2} = \frac{3}{2} \cdot n \log n$$

when n is a power of two, concluding the inductive step.

5. Let G be the following graph. The vertices of G are all the integers between -10 and 10 except for 0 (20 vertices in total). The pair $\{x, y\}$ is an edge of G if (and only if) $-30 < xy < 0$.

(a) Show that G is bipartite.

Solution: G is bipartite with respect to the partition N, P with $N = \{-10, \dots, -1\}$ and $P = \{1, \dots, 10\}$: If x and y are both in P or both in N then $xy > 0$ so $\{x, y\}$ is not an edge of G.

(b) Show that G does not have a perfect matching.

Solution: To show that G has no perfect matching, we exhibit a subset S of P of size 5 whose neighbour set has size at most 4. By Hall's theorem, the vertices in S cannot all be matched. Take $S = \{6, 7, 8, 9, 10\}$. The vertices -5, -6, -7, -8, -9 and -10 are not neighbours of S since any product between one of this numbers and a number in S is at most $-5 \cdot 6 = -30$. Therefore S can have at most 4 neighbours.

6. A *cut-edge* in a connected graph is an edge e such that if e was removed, the graph would no longer be connected. Show that any connected graph in which all vertices have even degree does not have a cut-edge.

Solution: We prove this is impossible by contradiction. Suppose there exists a connected graph G with exactly one cut-edge $e = \{u, v\}$ in which all vertices have even degree. Let G' be the graph obtained by removing the edge e from G. Then u and v must belong to different connected components C and C' of G' (for otherwise removing e from G would not disconnect it.) All the vertices of C except for u have even degree, so C has exactly one vertex of odd degree. Therefore the sum of the degrees of the vertices in C is odd. This is impossible: In lecture 5 we showed that the sum of the degrees of all vertices in a graph (and therefore in all of its connected components) must be even.

Practice Final 3

1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".

Solution: $\forall u, b_1, b_2\colon IN(b_1, u) \text{ AND } IN(b_2, u) \longrightarrow b_1 = b_2$. Any two balls in any given urn must be the same ball.

2. The sequence $f(n)$ is given by $f(n+1) = 2^{f(n)}$ for $n \geq 1$ with $f(0) = 2$.

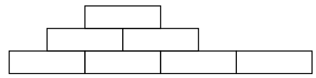
(a) Calculate $f(n) \bmod 5$ for $n = 1$, $n = 2$, and $n = 3$.

Solution: $f(1) = 2^2 = 4$, $f(2) = 2^{f(1)} = 16$ so $f(2) \equiv 1 \pmod 5$, $f(3) = 2^{16}$. This is a pretty large number but we can calculate $2^{16} \pmod 5$ using Fermat's little Theorem: $2^{16} \equiv (2^4)^4 \equiv 1^4 \equiv 1 \pmod 5$.

(b) Give a formula for $f(n)$ mod 5 for all $n \geq 4$. Justify your answer.

Solution: $2^k \equiv 1 \pmod 5$ whenever k is a multiple of 4. As $f(n)$ is a multiple of 4 for every $n \geq 1$ (it is a product of many 2s) we get that we get that $f(n+1) = 2^{f(n)} \equiv 1 \pmod 5$ for all $n \geq 1$.

3. Blocks of height one are stacked in layers in some formation. Each layer has strictly fewer blocks than the one under it. For example the 7-block formation below has height 3. Show that the height of an n-block formation is $O(\sqrt{n})$.



Solution: A formation of height k must have at least i blocks in its i-th level from the top because the number of blocks increases by one in each level. Therefore the number of blocks n must be at least $1 + 2 + \cdots + k = k(k+1)/2 > k^2/2$. Therefore $k < \sqrt{2n}$ which is $O(\sqrt{n})$.

4. You drop 30 balls into 7 urns. Some of the balls are red and some are blue. Show that at least three balls of the same colour land in the same urn.

Solution: Let $f\colon \{1, \ldots, 30\} \to \{1, \ldots 7\} \times \{\text{red}, \text{blue}\}$ be the function that assigns each ball to its urn and its colour. The domain of f has size 30 and its range has size $7 \cdot 2 = 14$. Since $30 > 2 \cdot 14$, by the generalized pigeonhole principle there exist 3 balls that are assigned to the same urn and have the same colour.

5. G is a directed graph whose vertices are the integers from -10 to 10 (inclusive) and whose edges (x, y) are those ordered pairs for which $|x| - |y| = 1$. For each of the following claims, say if it is true or false and provide a proof.

(a) G has a path of length 10.

Solution: True. The path $(10, 9, 8, \ldots, 0)$ has length 10.

(b) G has a parallel schedule of duration 11.

Solution: True. G cannot have a path of length 11 because the absolute value of vertices starts at 10 and must decrease along any path. As G has a parallel schedule whose length is the maximum path length plus one it must have a parallel schedule of size 11.

(c) G has an antichain of size 6.

Solution: True. The set $\{-10, 10, -8, 8, -6, 6\}$ is one such antichain.

6. The vertices of graph H_n are the n integers from $-n$ to n except 0. The edges of H_n are the pairs $\{x, y\}$ such that $x = -y$ or $|y - x| = 1$.

(a) Show that H_n is bipartite.

Solution: Let A be the union of even positive and odd negative vertices and B be the union of even negative and odd positive vertices. There are no edges within A and no edges between B.

(b) How many perfect matchings do H_1 and H_2 have?

Solution: H_1 consists of a single so it has one perfect matching. H_2 is a cycle of length 4 so it has two, namely $\{\{-1, 1\}, \{-2, 2\}\}$ and $\{\{-2, -1\}, \{1, 2\}\}$.

(c) How many perfect matchings does H_{10} have? (**Hint:** Write a recurrence.)

Solution: Let $f(n)$ denote the number of matching of the analogous graph H_n with $2n$ vertices in which the integers -10 and 10 are replaced by $-n$ and n. There are two possible ways in which vertex n can be matched: Either it is matched to $-n$, in which case the remaining vertices to be matched induce the graph H_{n-1}, or it is matched to $n - 1$, in which case $-n$ must also be matched to $-(n + 1)$ and the remaining vertices to be matched induce the graph H_{n-2}. Therefore the number of matchings $f(n)$ satisfies the recurrence $f(n) = f(n - 1) + f(n - 2)$ for all $n \geq 2$ with $f(1) = 1$ and $f(2) = 2$. This is the Fibonacci recurrence and we can calculate the following values for $f(n)$ when $n \leq 10$:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	2	3	5	8	13	21	34	55	89

so $f(10) = 89$.

Practice Final 4

1. Let $P(n)$ be the statement "There exists an $n \times n$ table of numbers in which the sum of every row is even and the sum of every column is odd".

   (a) Prove that $P(2)$ is true. Specify your proof method.

   Solution: We prove it by constructing an example:

   ```
   1 1
   2 2
   ```

   (b) Prove that $P(3)$ is false. Specify your proof method.

   Solution: We prove this by contradiction. Suppose such a 3×3 table exists. Let S be the sum of all nine entries. S can be obtained by adding the three row-sums, all of which are even, so it must be even. It can also be obtained by adding the three column-sums, all of each are odd, so it must also be odd. Contradiction.

2. Let $B_0 = 0$, $B_1 = 1$, and $B_n = B_{n-1} + \frac{1}{2}B_{n-2}$ for all $n \geq 2$. Prove that $B_n \geq ((1+\sqrt{3})/2)^{n-2}$ for all $n \geq 1$. Specify your proof method.

   Solution: We prove the inequality by strong induction on n. In the base case $n = 1$, $((1+\sqrt{3})/2)^{-1} = 2/(1+\sqrt{3}) \leq 2/(1+2) \leq 1$ as desired. When $n = 2$, $B_2 = B_1 + \frac{1}{2}B_0 = 1$ which equals $((1+\sqrt{3}/2)^0$ as desired again. Now assume the inequality is true for $n-1$ and $n-2$. Then

$$\begin{aligned} B_n &= B_{n-1} + \frac{1}{2}B_{n-2} \\ &\geq \left(\frac{1+\sqrt{3}}{2}\right)^{n-3} + \frac{1}{2}\left(\frac{1+\sqrt{3}}{2}\right)^{n-4} \\ &= \left(\frac{1+\sqrt{3}}{2}\right)^{n-4}\left(\frac{1+\sqrt{3}}{2} + \frac{1}{2}\right) \\ &= \left(\frac{1+\sqrt{3}}{2}\right)^{n-4}\left(\frac{1+\sqrt{3}}{2}\right)^{2} \\ &= \left(\frac{1+\sqrt{3}}{2}\right)^{n-2}. \end{aligned}$$

   In the second step we applied the identity

$$\left(\frac{1+\sqrt{3}}{2}\right)^2 = \frac{1+2\sqrt{3}+3}{4} = \frac{1+\sqrt{3}}{2} + \frac{1}{2}.$$

3. Find integers x and y between 0 and 16 that satisfy the following equations. Justify your steps.

$$\begin{aligned} 4x + 3y &\equiv 2 \pmod{17} \\ 3x + 4y &\equiv 3 \pmod{17}. \end{aligned}$$

   Solution: After eliminating x we get $7y \equiv 6 \pmod{17}$. To find the multiplicative inverse of 7 we apply extended Euclid's algorithm:

$$\begin{aligned} E(17,7) &= E(7,3) && 17 = 2 \cdot 7 + 3 \\ &= E(3,1) && 7 = 3 \cdot 2 + 1 \\ &= E(1,0) \end{aligned}$$

from where

$$1 = 7 - 2 \cdot 3 = 7 - 2 \cdot (17 - 2 \cdot 7) = -2 \cdot 17 + 5 \cdot 7.$$

Therefore $7^{-1} \equiv 5 \pmod{17}$ and $y \equiv 6 \cdot 5 \equiv 13 \pmod{17}$.

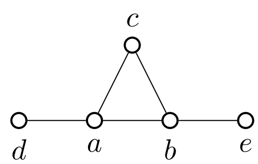
We can solve for x by substituting in the first equation but this would require taking another inverse. Instead if we subtract the equations we get $x - y \equiv -1 \pmod{17}$ so $x \equiv y - 1 \equiv 12 \pmod{17}$.

4. Does there exist a graph G with five vertices a, b, c, d, e such that

(a) The degrees of the five vertices are

v	a	b	c	d	e
deg v	3	3	2	1	1

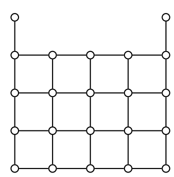
Solution: Yes, for example



(b) The degrees are as in part (a) and G is bipartite?

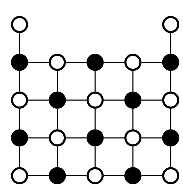
Solution: No. We prove that this is impossible by contradiction. In a bipartite graph (S, T) the sum of the degrees in S and T must be the same, namely $(3+3+2+1+1)/2 = 5$. Therefore a and b must be assigned to opposite sides of the partition, say a to S and b to T. As one of the sets, say T, will have size at most two, a cannot have three neighbors, a contradiction.

5. Let G be the following graph.



(a) Show that G is bipartite.

Solution: The following partition of the vertices into White and Black demonstrates that G is bipartite as there are no edges within W or within B.



(b) Does G have a perfect matching? Justify your answer.

Solution: No. There are 10 vertices in B and 12 in W, so the vertices in W cannot all be matched.

6. Alice, Bob, and Charlie play a game. Initially Alice holds \$1, Bob holds \$2, and Charlie holds \$5. In each round every player splits their holdings evenly in two and gives them away to the other two players.

(a) Let $a(n)$ be Alice's holdings after n rounds. Calculate $a(1)$ and $a(2)$.

Solution: $a(1) = (2+5)/2 = 7/2$. To calculate $a(2)$ we need to know how much Bob and Charlie hold in round 1: Bob holds $b(1) = (1+5)/2 = 3$ and Charlie holds $c(1) = (1+2)/2 = 3/2$. Then $a(2) = (3 + 3/2)/2 = 9/4$.

(b) Show that $a(n+1) = 4 - a(n)/2$. (**Hint:** The sum of all players' holdings remains invariant.)

Solution: For every n, $a(n) + b(n) + c(n)$ must equal 8. Therefore $a(n+1) = (b(n) + c(n))/2 = (8 - a(n))/2 = 4 - a(n)/2$.

(c) Solve the recurrence from part (b) with initial condition $a(0) = 1$ by unfolding or homogenization.

Solution: We unwind the recurrence:

$$\begin{aligned}
a(n) &= -\frac{1}{2}a(n-1) + 4 \\
&= -\frac{1}{2}\Big(-\frac{1}{2}a(n-2) + 4\Big) + 4 = \Big(-\frac{1}{2}\Big)^2 a(n-2) - \frac{1}{2}\cdot 4 + 4 \\
&= \Big(-\frac{1}{2}\Big)^2\Big(-\frac{1}{2}a(n-3) + 4\Big) - \frac{1}{2}\cdot 4 + 4 = \Big(-\frac{1}{2}\Big)^3 a(n-3) + \Big(-\frac{1}{2}\Big)^2\cdot 4 - \frac{1}{2}\cdot 4 + 4 \\
&= \cdots \\
&= \Big(-\frac{1}{2}\Big)^n a(0) + \Big(-\frac{1}{2}\Big)^{n-1}\cdot 4 + \Big(-\frac{1}{2}\Big)^{n-2}\cdot 4 + \cdots + 4.
\end{aligned}$$

Plugging in $a(0) = 1$ and using the geometric sum formula we get

$$a(n) = \Big(-\frac{1}{2}\Big)^n + 4\cdot\frac{1 - (-1/2)^n}{3/2} = \frac{8}{3} - \frac{5}{3}\cdot\Big(-\frac{1}{2}\Big)^n.$$

This evaluates to $a(0) = 1, a(1) = 7/3, a(2) = 9/4$ which is consistent with part (a).

Alternative solution: We try the homogenization $a(n) = a'(n) + c$ to get $a'(n+1) + c = 4 - (a'(n) + c))/2 = 4 - a'(n)/2 - c/2$, from where c must equal $8/3$. We solve $a'(n)$ to

$$a'(n) = (-1/2)a'(n-1) = (-1/2)^2 a'(n-2) = \cdots = (-1/2)^n a'(0) = (-1/2)^n \cdot (-5/3).$$

Therefore $a(n) = 8/3 - (5/3)(-1/2)^n$.