Practice Midterm 1

1. Are the propositions "Every two people have a common friend" and "Every person has at least two friends" logically equivalent? Justify your answer.

2. Alice has infinitely many \$6, \$10, and \$15 stamps. Can she make all integer postages of \$30 and above?

3. Prove that for every integer n there exists an integer k such that $|n^2 - 5k| \leq 1$. (**Hint:** What is $n^2 \bmod 5$?)

4. The numbers `12345678` are listed in order. In each step you can take three consecutive numbers abc and reorder them as cab, for example `2543`$\underline{\texttt{168}}$`7` $\rightarrow$ `2543`$\underline{\texttt{816}}$`7`. Can you ever obtain `81234567`?

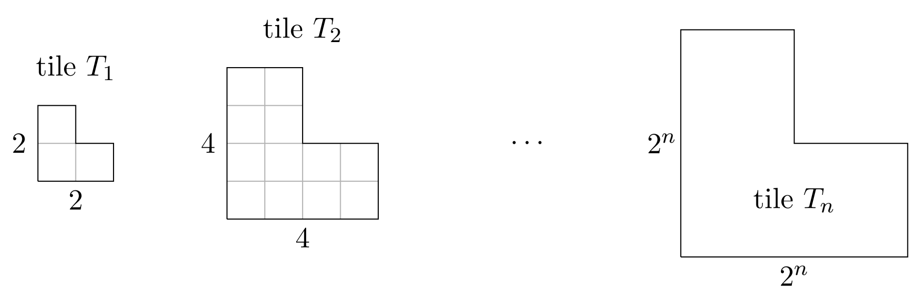
Practice Midterm 2

1. Express the sentence "Any two people who are not friends have a friend in common" using quantifiers and logical operators. Use x, y, z as variables and $F(x, y)$ for "x and y are friends."

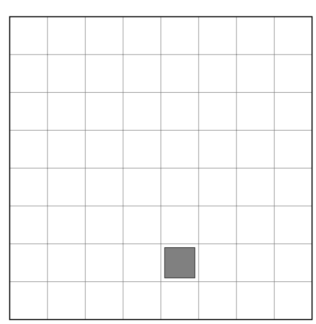
2. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

3. Can 4 be expressed as an integer linear combination of 47 and 13? If no, provide a proof. If yes, give such a combination and explain how you obtained it.

4. **Claim:** For every $n \geq 1$, a $2^n \times 2^n$ board with one square removed (in any position) can be filled with tiles $T_1, \ldots, T_n$ below (one of each type).



   (a) Describe a tiling for the following board ($n = 3$ with square $(5, 2)$ missing).



   (b) Prove the claim. Specify your proof method.

Practice Midterm 3

1. Underline and explain the mistake in the following "proof."

   Theorem. In every group of friends there exists a person with an even number of friends.

   Proof. By induction on the number of people n. When $n = 1$ the one person has zero friends, and zero is even. Now assume it is true for groups of n people. Let G be a group of $n+1$ people. Take out any person from G. By inductive hypothesis the remaining group G' has someone, say Alice, with an even number of friends. Since Alice is also in G, G has a person with an even number of friends. □

2. Prove that for every positive integer n, $\gcd(n^2 + n + 1, n + 1) = 1$.
   (**Hint:** Use the connection between gcd and combinations.)

3. For which nonzero integers n is the number $\frac{\sqrt{2}}{n} - \frac{n}{\sqrt{2}}$ rational? Justify your answer.

4. Bob has 32 *b*lue, 33 *r*ed, and 34 *g*reen balls. At every turn he takes out two balls and replaces them with two different balls by the following replacement rule:

   $$bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb.$$

   (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

   (b) Can Bob obtain 99 balls of the same color? Justify your answer.
   (**Hint:** Look at the difference between the number of red and blue balls.)

Practice Midterm 4

1. Is the following deduction rule valid?

   $$\frac{\forall x \exists y\colon P(x,y) \qquad \exists x \forall y\colon P(x,y)}{\forall x \forall y\colon P(x,y)}$$

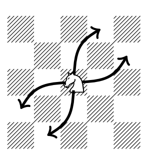
2. Show that for every positive real number x, at least one of the numbers $\sqrt{x} + 1$ and $\sqrt{2} \cdot x$ is irrational.

3. Bob has received from Alice the RSA ciphertext $c = 2$. The modulus is $n = pq$ with $p = 3$ and $q = 5$. The encryption key is $e = 3$.

   (a) Calculate Bob's decryption key d.

   (b) Decrypt Alice's message m.

4. A knight jumps around an infinite chessboard. Owing to injury it can only make these four moves:

   (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

   (b) Can the knight ever reach the square immediately to the left of its initial one?