

Alice, Bob, and Charlie hold 10 tokens each. They take turns flipping a coin.

- If a player flips heads their holdings are tripled.
  For example, if Alice had 5 tokens and flipped heads she will now have 15.
- If a player flips tails they must donate all their tokens to the other two (in any way they choose).
  In the same example, if Alice flipped tails she can choose to give 1 to Bob and 4 to Charlie.

(a) Prove that "the sum of all players' holdings is even" is an invariant.

Solution: In the start state the sum $a+b+c$ of the players' holdings is $3 \cdot 10 = 30$ which is even. Now assume $a+b+c$ is even before a transition.

- If a head is flipped, one of the players' holdings, say a, triples, so $a+b+c$ increases by $2a$ which is even and the sum $3a+b+c$ remains even.
- If a tail is flipped, $a+b+c$ remains the same so it stays even.

(b) Can they reach a state in which each holds exactly 99 tokens? Justify your answer.

Solution: No. $99+99+99 = 3 \cdot 99$ is an odd number. The invariant does not hold in this state so it is unreachable.