Let

$$S(n) = 1 + n + n^2 + n^3 + \cdots + n^{n-1}.$$

Circle one of the following alternatives:

$\boxed{S \text{ is } o(n^n)}$ $\qquad$ n^n is $o(S)$ $\qquad$ S is $\Theta(n^n)$

Justify your answer.

Solution: By the geometric sum formula,

$$S(n) = \frac{n^n - 1}{n - 1}$$

When $n \geq 2$, $n^n - 1 \leq n^n$ and $n - 1 \geq n/2$ and so

$$S(n) \leq \frac{n^n}{n/2} = 2n^{n-1} = \frac{2}{n} \cdot n^n$$

which is $o(n^n)$ as $2/n$ is smaller than any fixed constant when n is large.

Alternative solution: Each of the first $n - 1$ terms of $S(n)$ is not greater than n^{n-2}. Therefore

$$S(n) \leq (n-1)n^{n-2} + n^{n-1} \leq 2n^{n-1}$$

is $O(n^{n-1})$, so it is $o(n^n)$.