Find the exact closed-form solution to the recurrence

$$f(n) = \frac{2}{3} f(n-1) + 2, \qquad f(0) = 0.$$

Solution: We unwind the formula for $f(n)$:

$$\begin{aligned} f(n) &= \tfrac{2}{3} f(n-1) + 2 \\ &= \tfrac{2}{3}\big(\tfrac{2}{3} f(n-2) + 2\big) + 2 = (\tfrac{2}{3})^2 f(n-2) + \tfrac{2}{3} \cdot 2 + 2 \\ &= (\tfrac{2}{3})^2 \big(\tfrac{2}{3} f(n-3) + 2\big) + \tfrac{2}{3} \cdot 2 + 2 = (\tfrac{2}{3})^3 f(n-3) + 2\big((\tfrac{2}{3})^2 + \tfrac{2}{3} + 1\big). \end{aligned}$$

Continuing in this manner for n steps we get

$$f(n) = (\tfrac{2}{3})^n f(0) + 2\big(1 + \tfrac{2}{3} + (\tfrac{2}{3})^2 + \cdots + (\tfrac{2}{3})^{n-1}\big).$$

By the initial condition $f(0) = 0$ and the geometric sum formula, we have

$$f(n) = 2 \cdot \frac{1 - (\frac{2}{3})^n}{1 - \frac{2}{3}} = 6\big(1 - (\tfrac{2}{3})^n\big) = 6 - 6(\tfrac{2}{3})^n.$$

Alternative solution: We homogenize the recurrence to $f(n) + c = \frac{2}{3}(f(n-1) + c)$ for some constant c. For the original recurrence to hold it must be that $2 + c = \frac{2}{3}c$ which solves to $c = -6$. Therefore

$$f(n) - 6 = \tfrac{2}{3}(f(n-1) - 6) = (\tfrac{2}{3})^2 (f(n-2) - 6) = \cdots = (\tfrac{2}{3})^n (f(0) - 6).$$

As $f(0)$ equals 0 we get $f(n) = 6 - 6 \cdot (\frac{2}{3})^n$.