1. Which of the following pairs of statements are logically equivalent?

(a) $P \longrightarrow (\text{NOT } Q)$
$Q \longrightarrow (\text{NOT } P)$

Solution: The two are logically equivalent:

P	Q	NOT P	NOT Q	$P \longrightarrow (\text{NOT } Q)$	$Q \longrightarrow (\text{NOT } P)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

(b) If it is raining then the ground is wet.
If it is not raining then the ground is not wet.

Solution: Let R and W be the propositions “it is raining” and “the ground is wet”, respectively. We want to know whether

$$R \longrightarrow W \qquad \text{is equivalent to} \qquad (\text{NOT } R) \longrightarrow (\text{NOT } W).$$

R	W	$R \longrightarrow W$	NOT R	NOT W	NOT $R \longrightarrow$ NOT W
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

They are not logically equivalent: In the case it is raining but the ground is not wet, the first one is false but the second one is true. We can say more: Neither one implies the other.

(c) If it is raining then the ground is wet.
It is not raining or the ground is wet.

Solution: Let R and W be the propositions “it is raining” and “the ground is wet”, respectively. We want to know whether

$$R \longrightarrow W, \qquad \text{is equivalent to} \qquad (\text{NOT } R) \text{ OR } W$$

The truth table is

R	W	$R \longrightarrow W$	$(\text{NOT } R)$ OR W
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Again, the two are logically equivalent.

(d) Not all three balls are of the same colour. (Colours are black or white.)
Among the three balls, there is exactly one white ball, or there is exactly one black ball.

Solution: They are logically equivalent. Let W_1 be the proposition “ball 1 is white” and so on. The first proposition is NOT $(W_1 \text{ IFF } W_2 \text{ IFF } W_3)$. The second one is a bit tedious to write in propositional logic. Instead of doing that let W and B be the propositions “exactly one of W_1, W_2, W_3 is true” and “exactly one is false”, respectively. The truth-table is

W_1	W_2	W_3	W_1 IFF W_2 IFF W_3	NOT (W_1 IFF W_2 IFF W_3)	W	B	W OR B
T	T	T	T	F	F	F	F
T	T	F	F	T	F	T	T
T	F	T	F	T	F	T	T
T	F	F	F	T	T	F	T
F	T	T	F	T	F	T	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	F	T
F	F	F	T	F	F	F	F

2. The Australian Open is a tennis tournament that is played in elimination rounds. 64 entrants are matched up to play, then so are the 32 winners of those matches, and so on until a single winner remains. $B(x, y)$ means that player x beat player y and $W(x)$ means that player x won the tournament. Translate the following propositions into plain English. Which are true?

   (a) $\exists x, y : B(x, y)$ AND $W(y)$

   Solution: There is a player (y) who was beaten (by x) but won the tournament. False: player y must have been eliminated.

   (b) $\forall x \exists y : B(x, y)$ OR $B(y, x)$

   Solution: Every player was either beaten by someone, or beat someone. True.

   (c) $\exists x, y, z : B(x, y)$ AND $B(y, z)$ AND $B(z, x)$

   Solution: There are three players out of which each one beat one of the other two and was beaten by the other. False: In an elimination tournament no three players get to all play one another.

   (d) $\forall x : (\exists y : B(y, x))$ IFF (NOT $W(x)$)

   Solution: A player doesn't win *if and only if* they were beaten by someone. True. If a player doesn't win they must have been beaten. Conversely, if a player was beaten they didn't win.

   (e) $\exists x : \forall y : B(x, y) \longrightarrow \exists z : B(y, z)$

   Solution: There is a player (x) such that anyone they beat (y) beat someone else (z). True: Any x who lost in the first round is such a player. As x didn't beat anyone, anything goes for every player y that they beat, including the claim that y beat someone else.

3. Express the following propositions about people and their relative heights using quantifiers and logical operators. Use x, y, z as variables and $T(x, y)$ for "x is taller than y". Make sure that all your variables are quantified. Explain your answer.

   (a) Bob is not the tallest person and he is not the shortest person.
   (b) One cannot be both taller and shorter than someone.
   (c) There is a shortest person, but there is no tallest person.
   (d) There are at least two people that are taller than Bob.
   (You can use $x \neq y$ for "x and y are different people".)

   Solution:

   (a) "Bob is not the tallest person" means there exists some person x that is taller than Bob. "Bob is not the shortest person" means there exists some person y that Bob is taller than.

   $$(\exists x\colon T(x, \text{Bob})) \text{ AND } (\exists y\colon T(\text{Bob}, y)).$$

   We could also write "there exist two people x and y so that Bob is shorter than x and taller than y:

   $$\exists x, y\colon T(x, \text{Bob}) \text{ AND } T(\text{Bob}, y).$$

   The two propositions are logically equivalent. Both answers are acceptable.

(b) Let's call "one" and "someone" x and y. This proposition says that it cannot be that x is taller than y and x is shorter than y:

$$\forall x, y \colon \text{NOT } (T(x,y) \text{ AND } T(y,x)).$$

Equivalently, you could also say there exists no pair x and y so that each is taller than the other:

$$\text{NOT } \exists x, y \colon T(x,y) \text{ AND } T(y,x).$$

The two propositions are logically equivalent and both answers are acceptable.

(c) "There is a shortest person" means there is some person, call him x, so that everyone is taller than x: In symbols, $\exists x \forall y \colon T(y,x)$. "There is no tallest person" means that for every person x, there is a y that is taller than him: $\forall x \exists y \colon T(y,x)$. The proposition in question is the and of these two:

$$(\exists x \forall y \colon T(y,x)) \text{ AND } (\forall x \exists y \colon T(y,x)).$$

(d) This proposition says that there are two distinct people x and y, both taller than Bob.

$$\exists x, y \colon T(x, \text{Bob}) \text{ AND } T(y, \text{Bob}) \text{ AND } (x \neq y).$$

4. The following propositions are about students and the exams they pass: $E(s,c)$ means "Student s passed the exam in course c", and $P(s)$ means "Student s won a prize".

(a) Explain the meaning of these two propositions in plain English:
$A \colon \forall m\ \exists s \colon E(s,m) \text{ AND } (\text{NOT } P(s))$
$B \colon \exists m\ \forall s \colon E(s,m) \longrightarrow P(s)$

(b) Can both A and B be true? Justify your answer.

(c) Explain the meaning of these two propositions in plain English:
$C \colon \forall m\ \exists s\ \forall n \colon E(s,m) \text{ AND } ((m \neq n) \longrightarrow (\text{NOT } E(s,n)))$
$D \colon \forall m, n\ \exists s \colon E(s,m) \text{ AND } ((m \neq n) \longrightarrow (\text{NOT } E(s,n)))$

(d) Are C and D logically equivalent? Justify your answer.

Solution:

(a) Proposition A says that for every subject, there is a student who passes the exam but doesn't win a prize. Proposition B says that there is a subject for which everyone that passes the exam wins a prize.

(b) No. If proposition B is true, then there is a subject, say Data Structures, so that everyone who passes this exam wins a prize. But then proposition A is false when m is Data Structures: There is no student who passes this exam but doesn't win a prize.

(c) Proposition C means "For each subject, there is a student that passes only the exam of that subject."
Proposition D means "For every two subjects, there exists a student that passes the exam in the first one but not in the second one."

(d) Proposition C implies proposition D, but D does not imply C. Thus they are not equivalent.
To see that D does not imply C, we describe a possible world in which D is true but C is false. Suppose there are three students Alice, Bob, and Charlie, and three subjects: math, algorithms, and programming. Suppose Alice only passes math and algorithms, Bob only passes programming and algorithms, and Charlie only passes programming and math. Then proposition D is true: For every pair of subjects, there is a student who passes one but not the other. But proposition C is false. The meaning of its negation NOT C is that there exists a subject which all students that passed also passed some other subject. One such subject is math: Alice and Charlie both passed math but also passed algorithms and programming, respectively.