1. Find and explain the mistakes in the following "proofs".

(a) **Theorem:** In every group of 5 people there is a person who is friends with at least 3 of them.

Proof. Let A and B denote two of the five people. The proof is by case analysis. We consider two cases:

- **Case 1:** A is friends with at least 3 other people in the group.
- **Case 2:** B is friends with at least 3 other people in the group.

It follows that at least one of A and B is friends with at least 3 other people, so a person that is friends with at least 3 others always exists. □

Solution: The two cases does not exhaust all possibilities. Recall the deduction rule for proof of proposition P by cases C_1 and C_2:

$$\frac{C_1 \text{ OR } C_2 \qquad C_1 \longrightarrow P \qquad C_2 \longrightarrow P}{P}$$

Here, C_1 OR C_2 is false: It could be that neither A nor B is friends with at least 3 others.

(b) **Theorem:** Every group of 8 people includes a group of 4 friends or a group of 4 strangers.

Proof. Let A be one of the eight people. The proof is by case analysis. We consider two cases:

- **Case 1:** A is friends with at least 4 other people in the group.
- **Case 2:** A is a stranger to at least 4 other people in the group.

One of these two cases must hold. Let's discuss Case 1. If all the people who are friends with A are strangers among themselves, this is a group of 4 strangers. Otherwise, at least 3 of them are mutual friends, and together with A they form a group of 4 friends.

Now let's do Case 2. If all the people who are strangers to A are friends among themselves, this is a group of 4 friends. Otherwise, at least 3 of them are mutual strangers, and together with A they form a group of 4 strangers. □

Solution: One problem in the proof is that an invalid deduction is made within the Case 1 analysis:

> If all the people who are friends with A are strangers among themselves, then [...]. Otherwise, at least 3 of them are mutual friends.

Let P be the proposition "All people who are friends with Alice are strangers among themselves" and Q be the proposition "At least 3 of them are mutual friends." The "proof" makes the deduction

$$\frac{\text{NOT } P}{Q}$$

This deduction is incorrect: NOT P is the proposition "There exists a pair of friends among the friends of A". This does not allow us to deduce Q: It could be that among the friends of A, there exists exactly one pair of friends. Then NOT P would be true but Q would be false, so the deduction is invalid.

Similar problem occurs in the analysis of Case 2.

(c) **Theorem:** In every 3 by 3 table containing the digits `1` to `9` each once, some two consecutive digits must appear in the same row or in the same column.

Proof. We prove this by contradiction. Once we fix the placement of 1 there are four positions from which 2 is blocked, namely the two in the same column and the two in the same row. When we position 2 in one of the remaining ones, there are now four more positions from which 3 is blocked. We repeat the argument one more time. There are now a total of $3 \times 4 = 12$ blocked positions so 4 cannot be placed anywhere in the table. This is a contradiction so a table with the desired properties cannot exist. □

Solution: The mistake in this proof is that it assumes (without saying so explicitly) that the positions from which number n is blocked includes all positions from which numbers 2 up to $n-1$ were blocked. This is already false for $n = 3$. The positions from which 3 is blocked only include those in the rows and columns of 2 and not the ones in the rows and columns of 1.

2. Prove the following theorems using the specified proof method.

   (a) If a is even or b is even then $a^2 \cdot b$ is even. (Cases)

   Solution: If a is even then $a = 2k$ for some integer k so $a^2 b = 4kb$ is an even number. If b is even then $b = 2\ell$ for some integer ℓ so $a^2 b = 2a^2\ell$ is an even number.

   (b) If $a^2 \cdot b$ is even then a is even or b is even. (Contrapositive)

   Solution: We prove this by contrapositive. Assume a and b are both odd. In class we showed that the product of odd numbers is odd, so $a^2 b$ is also odd.

   (c) Any 3×3 table containing each of the numbers 1 to 9 exactly once has (at least) two even numbers in the same row. (Contradiction)

   Solution: Assume for contradiction that every row in the table has at most one even number. Then the table can have at most three even numbers in it. There are four even numbers between 1 and 9 so such a table

3. Prove the following theorems. Specify your proof method.

   (a) For every odd integer n, $3n^2 - 7$ is a multiple of 4.

   Solution: We prove this by direct implication. Assume n is odd. We can write $n = 2k + 1$ for some k. Now, substitute $n = 2k + 1$ into $3n^2 - 7$:

$$\begin{aligned} 3n^2 - 7 &= 3(2k+1)^2 - 7 \\ &= 3(4k^2 + 4k + 1) - 7 \\ &= 12k^2 + 12k + 3 - 7 \\ &= 12k^2 + 12k - 4 \\ &= 4(3k^2 + 3k - 1). \end{aligned}$$

   which is a multiple of 4.

   (b) For every integer n, the number $n^3 - 3n + 2$ is even.

   Solution: We prove this by cases. When n is even, so are n^3 and $3n$, so $n^3 - 3n + 2$ is the sum of even numbers and therefore even. When n is odd, n^3 and $3n$ are products of odd numbers so they must be odd. Therefore $n^3 - 3n + 2$ is a sum of two odd and one even number so it is even.

   (c) For all positive real numbers x and y, if x is irrational, at least one of the numbers $x + y$, $x^2 + y^2$, x^2 is irrational.

   Solution: We will prove the contrapositive: For positive x and y, if $x+y$, x^2+y^2, and x^2 are all rational, then x is a rational number. We will use that addition, subtraction, multiplication, and division (not by zero) all preserve rationality of numbers.
   Assume $x + y, x^2 + y^2, x^2$ are all rational. Then so is $(y^2 + x^2) - 2x^2 = y^2 - x^2 = (y - x)(y + x)$. As $y + x$ is positive, we can divide by it and obtain that $y - x$ is rational. Then $(y + x) - (y - x) = 2x$ is also rational, and so must be x.

4. Which of these propositions are true and which are false? If a proposition is true, prove it. If it is false, prove its negation. (If you claim a number is irrational, provide a proof or give a reference, for example "By Theorem 9 in Lecture 2, $\sqrt{2}$ is irrational.")

   (a) If $-1 \leq x \leq 0$ then $x^3 + 3x^2 + x - 1 < 0$.

   Solution: False because $(-1)^3 + 3(-1)^2 + (-1) - 1 = 0$.

   (b) $\sqrt{3} + \sqrt{6}$ is an irrational number.

   Solution: True. We prove this by contradiction. Assume that it is a rational number r. Then

$$r^2 = (\sqrt{3} + \sqrt{6})^2 = 3 + 6 + 2\sqrt{3 \cdot 6} = 9 + 6\sqrt{2}.$$

   Therefore $\sqrt{2} = (r^2 - 9)/6$, which is a rational number, contradicting Theorem 9 from Lecture 2.

   (c) For all irrational numbers x, the number $x^2 - \sqrt{2}$ is irrational.

   Solution: False. Let $x = 1 - \sqrt{2}/2$. Then, $x^2 + \sqrt{2} = (1 - \sqrt{2}/2)^2 + \sqrt{2} = (1 - \sqrt{2} + 1/2) + \sqrt{2} = 3/2$ is a rational number. (*Alternative solution:* $x = \sqrt[4]{2}$ is the square root of $\sqrt{2}$ and therefore irrational, but $x^2 - \sqrt{2} = 0$ is rational.)

   (d) $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is an irrational number.

   Solution: True. For contradiction, assume the number $x_0 = \sqrt{2} + \sqrt{3} + \sqrt{5}$ is rational. Then the numbers

$$\begin{aligned}
x_1 &= (x_0^2 - 10)/2 &&= \sqrt{6} + \sqrt{10} + \sqrt{15} \\
x_2 &= (x_1^2 - 31)/2 &&= 5\sqrt{6} + 3\sqrt{10} + 2\sqrt{15} \\
x_3 &= x_2 - 3x_1 &&= 2\sqrt{6} - \sqrt{15} \\
x_4 &= (39 - x_3^2)/12 &&= \sqrt{10}
\end{aligned}$$

   are also rational.
   We prove $\sqrt{10}$ is irrational just like we proved $\sqrt{2}$ is irrational. If $\sqrt{10}$ can be expressed as n/d for integers n and $d \neq 0$ without a common factor, then $n^2 = 10d^2$ so n^2 is even and so is n. Writing $n = 2k$ we get that $4k^2 = 10d^2$ so $5d^2 = 2k^2$, so $5d^2$ is even. If d were odd, $5d^2$ would be a product of odd numbers and therefore odd, so d must be even itself. Then n and d have two as a common factor, a contradiction.