1. Prove the following using induction. n is a positive integer.

(a) For every n, the sum of the first n odd integers is n^2.

Solution: Let $S(n)$ be the sum of the first n odd positive integers. We prove that $S(n) = n^2$ for all $n \geq 1$ by induction on n.
Base case $n = 1$: $S(1) = 1$ and $1^2 = 1$, so the base case holds.
Inductive step: We assume that $S(n)$ equals n^2. The $(n+1)$-st odd positive integer is $2n+1$ and so

$$S(n+1) = S(n) + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

It follows by induction that $S(n) = n^2$ for all positive integers n.

(b) For every n, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Solution: Base case $n = 1$: $1/1^2 = 1 = 2 - 1/1$.
Inductive step: We assume that $1/1^2 + \cdots + 1/n^2 \leq 2 - 1/n$. By the inductive hypothesis

$$\frac{1}{1^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \left(\frac{1}{n} - \frac{1}{(n+1)^2}\right).$$

We will now show that the expression in the parenthesis is *at least* $1/(n+1)^2$, so that the right hand side is at most $2 - 1/(n+1)^2$ completing the inductive step.

$$\frac{1}{n} - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - n}{n(n+1)^2} = \frac{n^2+n+1}{n(n+1)^2} > \frac{n^2+n}{n(n+1)^2} = \frac{1}{n+1}.$$

(c) $(2n)!/(n!n!) \leq 4^n$. (**Optional:** $4^n/2n \leq (2n)!/(n!n!)$.)

Solution: Base Case: When $n = 1$, $4^1/2 = 2$, $2!/(1!1!) = 2$, and $4^1 = 4$ so both inequalities hold.
Inductive Step: We assume that

$$\frac{(2n)!}{n!n!} \leq 4^n$$

and deduce that

$$\frac{(2(n+1))!}{(n+1)!(n+1)!} = \frac{(2n)! \cdot (2n+1)(2n+2)}{n! \cdot (n+1) \cdot n! \cdot (n+1)} = \frac{(2n)!}{n!n!} \cdot \frac{2n+1}{n+1} \cdot \frac{2n+2}{n+1} \leq 4^n \cdot 2 \cdot 2 \leq 4^{n+1}$$

completing the inductive step.
For the optional part, the base case is $4^1/2 = 2 = 2!/(1!1!)$. For the inductive step we assume

$$\frac{4^n}{2n} \leq \frac{(2n)!}{n!n!}$$

and deduce that

$$\frac{4^{n+1}}{2(n+1)} = \frac{4^n}{2n} \cdot \frac{4n}{n+1} \leq \frac{(2n)!}{n!n!} \cdot \frac{4n}{n+1} = \frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot \frac{(n+1)^2}{(2n+1)(2n+2)} \cdot \frac{4n}{n+1}.$$

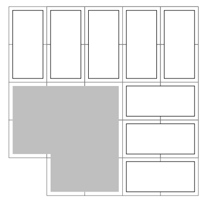
The product of the last two terms is at most one because it simplifies to $2n/(2n+1)$, completing the inductive step.

(d) For every odd n, an $n \times n$ grid with a corner square removed can be tiled using 2×1 pieces. (See example for $n = 3$. Your proof may describe a different tiling.)

Solution: Base case $n = 1$: A one by one grid with a square removed is a grid with no square, which can be tiled by zero 2×1 pieces.

Inductive step: We assume the predicate is true for n, where n is odd. We show that it is true for the next odd number $n+2$, namely an $(n+2) \times (n+2)$ with a corner square removed can be tiled with 2×1 pieces. Tile the first two rows of the grid with consecutive vertical 1×2 tiles. Tile what remains of the last two columns with horizontal 2×1 tiles. What remains to be tiled is a $n \times n$ subgrid with a corner square removed. This can be done by the inductive hypothesis.

The figure below illustrates the tiling when $n = 5$. The gray shape is tiled using the inductive hypothesis.



2. Use strong induction to prove the following for all positive integers $n \geq 1$.

(a) If $n \geq 14$ then n can be written as $4a + 5b$ for some integers $a, b \geq 0$.

(b) $(3/2)^{n-2} \leq F_n \leq (7/4)^{n-2}$ for every $n \geq 4$, where F_n is the n-th Fibonacci number ($F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$.)

(c) $G_n = n!$, where G_n is given by $G_1 = 1$, $G_{n+1} = 1 + G_1 + 2G_2 + \cdots + nG_n$. (**Hint:** $n \cdot n! = (n+1)! - n!$).

Solution:

(a) We first check that the predicate is true for $n = 14$, 15, 16 and 17:

$$\begin{aligned}14 &= 4 \cdot 1 + 5 \cdot 2\\ 15 &= 4 \cdot 0 + 5 \cdot 3\\ 16 &= 4 \cdot 4 + 5 \cdot 0\\ 17 &= 4 \cdot 3 + 5 \cdot 1.\end{aligned}$$

Now we prove that it is true for all $n \geq 14$ by strong induction. We assume that the predicate is true for all values from 14 up to n. If $n+1$ is 15, 16, or 17 it was already checked. If $n+1 > 17$, by our hypothesis we know that $(n+1) - 4 = 4a + 5b$ for some $a, b \geq 0$, so $n + 1 = 4(a+1) + 5b$.

(b) **Base case $n = 4$:** $(3/2)^{4-2} = 9/4 \leq F_4 = 3 \leq (7/4)^{4-2} = 49/16$.

Inductive step: Now assume that $(3/2)^{k-2} \leq F_k \leq (7/4)^{k-1}$ for k ranging from 1 up to n. It is easier to manage the two inequalities separately. For the upper bound,

$$F_{n+1} = F_n + F_{n-1} \leq (7/4)^{n-2} + (7/4)^{n-3} = (7/4)^{n-1}(4/7 + (4/7)^2) = (7/4)^{n-1} \cdot (44/49) \leq (7/4)^{n-1}.$$

The first inequality is our use of the inductive hypothesis. For the lower bound,

$$F_{n+1} = F_n + F_{n-1} \geq (3/2)^{n-2} + (3/2)^{n-3} = (3/2)^{n-1}(2/3 + (2/3)^2) = (3/2)^{n-1} \cdot (10/9) \geq (3/2)^{n-1}.$$

The first inequality is again our use of the inductive hypothesis.

(c) **Base case:** $n = 1$ $G_1 = 1 = 1!$. So the claim holds for $n = 1$.

Inductive step: Now, assume $G_n = n!$ for $k = 1, 2$ up to n. Let $k = n + 1$. Then

$$\begin{aligned}G_{n+1} &= 1 + G_1 + 2G_2 + \cdots + nG_n\\ &= 1 + 1! + 2(2!) + \cdots + n(n!)\\ &= 1 + (2! - 1!) + (3! - 2!) + \cdots + ((n+1)! - n!)\\ &= 1 - 1! + 2! - 2! + 3! + \cdots - n! + (n+1)!\end{aligned}$$

which equals $(n+1)!$ as desired because all other terms cancel out.

3. n white pegs and n black pegs are arranged in a line. In each step you are allowed to move any peg past *two* consecutive pegs of the opposite color, left or right. Initially all white pegs are to the left of the black ones.

$$\circ\,\circ\,\bullet\,\bullet\,\circ\,\bullet \longrightarrow \circ\,\bullet\,\bullet\,\circ\,\circ\,\bullet$$

(a) Assume n is odd. Say a pair of pegs is *inverted* if one is black, one is white, and the black one is to the left of the right one. Prove that "the number of inverted pairs is even" is an invariant.

Solution: This is initially true as the number of inverted pairs is zero. Now assume it is true after t steps. In step $t + 1$, the number of inverted pairs goes up by two if a white peg jumps to the right or a black peg jumps to the left, or down by two if a white peg jumps to the left or a black peg jumps to the right. In all cases, the number of inverted pairs stays even.

(b) If n is odd, can the colors be reversed so that all black pegs are to the left of all white ones?

$$\underset{\text{INITIAL}}{\circ\,\circ\cdots\circ\,\bullet\,\bullet\cdots\bullet} \qquad \underset{\text{FINAL}}{\bullet\,\bullet\cdots\bullet\,\circ\,\circ\cdots\circ}$$

Solution: No. In the final configuration, every one of the n^2 black-white pairs is inverted. Since n is odd, n^2 is also odd so there is an odd number of inverted pairs. Owing to the invariant from part (a) the final configuration can never be reached

(c) If n is even, can the colors be reversed?

Solution: Yes. More generally we show by induction on k that this is true for any number k of white pegs and n black pegs (as long as n is even). When $k = 0$ there are no white pegs so there is nothing to reverse. Now we assume k white pegs and n black pegs can be reversed. Given $k + 1$ white pegs and n black pegs, move the rightmost white peg to the right end by jumping two black pegs at a time and leave it there. By inductive hypothesis the remaining $k + n$ pegs can be reversed, so the whole configuration can be reversed.

4. You have a system of n switches, each of which can be in one of two states: OFF or ON. There are 2^n possible configurations of this system. For example, when $n = 2$ the four possible configurations for the pair of switches are (OFF, OFF), (OFF, ON), (ON, OFF), and (ON, ON).

Initially, all switches are OFF. In each step, you are allowed to flip exactly one of the switches. Is there a sequence of flips that makes each possible configuration arise exactly once? For example when $n = 2$ this sequence of flips has the desired property (the number on the arrow indicates the switch that is flipped):

$$(\text{OFF}, \text{OFF}) \xrightarrow{2} (\text{OFF}, \text{ON}) \xrightarrow{1} (\text{ON}, \text{ON}) \xrightarrow{2} (\text{ON}, \text{OFF})$$

(a) Show a sequence of flips that works when $n = 3$.

Solution: Here is one possible sequence of flips:

$$(\text{OFF}, \text{OFF}, \text{OFF}) \xrightarrow{3} (\text{OFF}, \text{OFF}, \text{ON}) \xrightarrow{2} (\text{OFF}, \text{ON}, \text{ON}) \xrightarrow{3} (\text{OFF}, \text{ON}, \text{OFF})$$
$$\xrightarrow{1} (\text{ON}, \text{ON}, \text{OFF}) \xrightarrow{3} (\text{ON}, \text{ON}, \text{ON}) \xrightarrow{2} (\text{ON}, \text{OFF}, \text{ON}) \xrightarrow{3} (\text{ON}, \text{OFF}, \text{OFF}).$$

The sequence of moves first covers all possible configuration of the last two switches with the first one ON, then flips the first one switch to OFF and again covers all possible configurations of the last two switches.

(b) Prove that for every $n \geq 1$, there exists a sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration. (**Hint:** Use induction. You may need to strengthen the proposition.)

Solution: We prove the following stronger proposition:

Theorem: for every $n \geq 1$, and *every possible starting configuration*, there exists a sequence of flips for n switches that covers every possible configuration exactly once.

Proof. We apply induction on n. For the base case $n = 1$, regardless of the starting configuration, we cover each of the two configurations exactly once by a single switch flip.

For the inductive step, we assume the predicate of interest holds for n. Now given any starting configuration of $n+1$ switches consider the following sequence of moves. First, by the inductive assumption we can cover all possible configurations of the last n switches without touching the first switch. Then we flip the first switch. Then again, by the inductive assumption we cover all possible configurations of the last n switches without touching the first one. Each configuration of the $n+1$ switches will then be covered exactly once. $\square$

(c) Now suppose that you flip not one but two switches at a time. Prove that for every $n \geq 2$, there is no sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration. (**Hint:** Use an invariant.)

Solution: We show that "the number of ON switches is even" is an invariant. This holds initially as all switches are off. Now assume it is true at any given state. The following three cases cover all possibilities:

- **Case 1:** Two switches are flipped OFF: Then the number of ON switches decreases by two so it stays even.
- **Case 2:** Two switches are flipped ON: Then the number of ON switches increases by two so it stays even.
- **Case 3:** One switch is flipped ON and one is flipped OFF: The number of ON switches does not change.

Therefore, the configuration in which the first switch is ON and all others are OFF can never be reached.