1. Find exact closed form expressions for the following sums. Explain how you discovered the expression and prove that it is correct.

(a) $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2$.

Solution: Let $S(n)$ be the value of the sum. We guess that $S(n)$ has the form $an^3 + bn^2 + cn + d$ for some unknown a, b, c, d. Plugging in $n = 0, 1, 2, 3$ we get

$$\begin{aligned} d &= S(0) = 1^2 = 1 \\ a + b + c + d &= S(1) = 1^2 + 3^2 = 10 \\ 8a + 4b + 2c + d &= S(2) = 1^2 + 3^2 + 5^2 = 35 \\ 27a + 9b + 3c + d &= S(3) = 84. \end{aligned}$$

Plugging in $d = 1$ in the other equations we obtain a system of three linear equations in three unknowns a, b, c. The unique solution is $a = 4/3, b = 4, c = 11/3$. We now prove by induction that $S(n)$ equals $\frac{4}{3}n^3 + 4n^2 + \frac{11}{3}n + 1$. As for the inductive step, assuming the formula is true for n, showing that it also holds for $n+1$ amounts to verifying the identity

$$\tfrac{4}{3}(n+1)^3 + 4(n+1)^2 + \tfrac{11}{3}(n+1) + 1 = \left(\tfrac{4}{3}n^3 + 4n^2 + \tfrac{11}{3}n + 1\right) + \left(2(n+1)+1\right)^2.$$

Alternative solution: $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = A - B$, where

$$A = 1^2 + 2^2 + \cdots + (2n+1)^2 = \tfrac{1}{3}(2n+1)^3 + \tfrac{1}{2}(2n+1)^2 + \tfrac{1}{6}(2n+1)$$

by Theorem 1 from Lecture 7

$$B = 2^2 + 4^2 + \cdots + (2n)^2 = 4(1^2 + 2^2 + \cdots + n^2) = \tfrac{4}{3}n^3 + \tfrac{4}{2}n^2 + \tfrac{4}{6}n$$

by the same theorem. After simplifying the expression $A - B$ we get that

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \tfrac{4}{3}n^3 + 4n^2 + \tfrac{11}{3}n + 1.$$

(b) $3^n + 3^{n+1} + 3^{n+2} + \cdots + 3^{2n}$.

Solution: We can factor out 3^n from all terms and use the geometric sum formula to obtain

$$3^n + 3^{n+1} + 3^{n+2} + \cdots + 3^{2n} = 3^n(1 + 3 + 3^2 + \cdots + 3^n) = 3^n \cdot \frac{3^{n+1} - 1}{2}.$$

Alternative solution: $3^n + 3^{n+1} + 3^{n+2} + \cdots + 3^{2n}$ is the difference $A - B$ of the following two geometric sums

$$A = 1 + 3 + 3^2 + \cdots + 3^{2n} = \frac{3^{2n+1} - 1}{3 - 1} = \frac{3^{2n+1} - 1}{2}$$

and

$$B = 1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}.$$

So we have

$$3^n + 3^{n+1} + 3^{n+2} + \cdots + 3^{2n} = A - B = \frac{3^{2n+1} - 3^n}{2} = 3^n \cdot \frac{3^{n+1} - 1}{2}.$$

(c) (**Optional**) $1/2 + 2/2^2 + 3/2^3 + \cdots + n/2^n$.
(**Hint:** Call this number S. Subtract S from $2S$ term by term.)

Solution: Call this number S. Then

$$2S = 1 + \frac{2}{2} + \frac{3}{2^2} + \cdots + \frac{n}{2^{n-1}}.$$

If we match the terms of $2S$ and S with the same denominators and subtract we obtain

$$\begin{aligned} 2S - S &= 1 + \left(\frac{2}{2} - \frac{1}{2}\right) + \left(\frac{3}{2^2} - \frac{2}{2^2}\right) + \cdots + \left(\frac{n}{2^{n-1}} - \frac{n-1}{2^{n-1}}\right) - \frac{n}{2^n} \\ &= 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} - \frac{n}{2^n}. \end{aligned}$$

The first n terms form a geometric sum with base $1/2$, so

$$S = 2S - S = \frac{1-(1/2)^n}{1-1/2} - \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

2. Show the following inequalities by using the integral method for approximating sums.

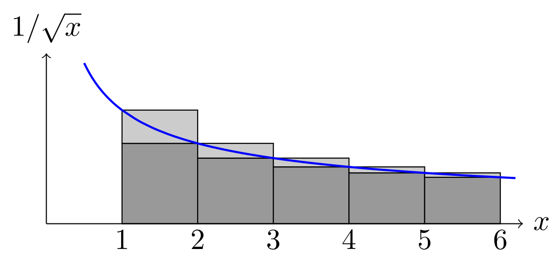
(a) $2\sqrt{n+1} - 2 \le 1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} \le 2\sqrt{n+1} - 1$.

(b) $n^3/3 \le 1^2 + 2^2 + \cdots + n^2 \le n^3/3 + n^2$.

(c) $1 \cdot e^{-1^2} + 2 \cdot e^{-2^2} + \cdots + n \cdot e^{-n^2} \le 3/(2e)$.

Solution:

(a) We approximate the sum by the integral of the function $f(x) = 1/\sqrt{x}$. The value of the sum from $1/\sqrt{1}$ to $1/\sqrt{n}$ equals the area under the first n bars in the following diagram.



The area is at least the integral of the function $f(x) = 1/\sqrt{x}$ from 1 to $n+1$. So, we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \ge \int_1^{n+1} \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_1^{n+1} = 2\sqrt{n+1} - 2.$$

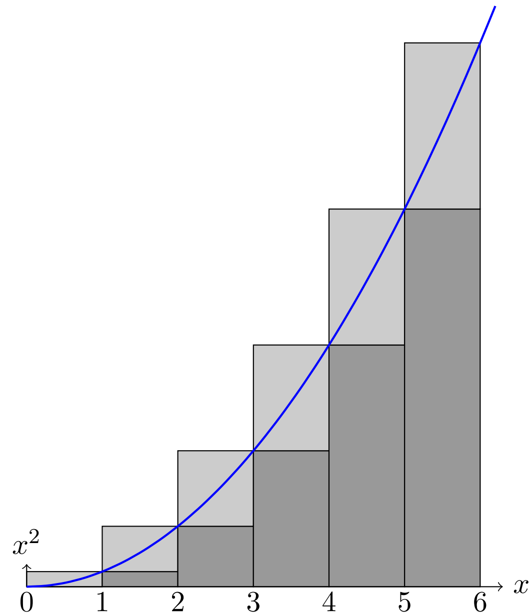
If we subtract the area of the light-shaded rectangles from the sum then the integral becomes an upper bound. The total area of the light-shaded rectangles is $1 - 1/\sqrt{n+1}$. Therefore

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - \left(1 - \frac{1}{\sqrt{n+1}}\right) \le \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx$$

from where we obtain the upper bound

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \le 2\sqrt{n-1} - 1 - \frac{1}{\sqrt{n+1}} \le 2\sqrt{n-1} - 1.$$

(b) We approximate the sum by the integral of the function $f(x) = x^2$. The sum equals the area under the first n bars in the following diagram.



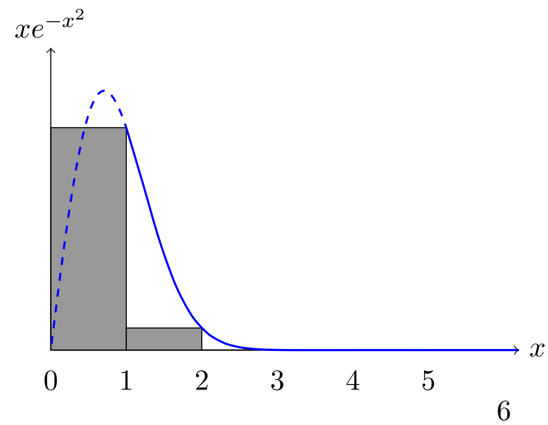
The area is at least the integral of the function $f(x) = x^2$ from 0 to $n+1$, so we have.

$$1^2 + 2^2 + 3^2 + ... + n^2 \geq \int_0^n x^2\, dx = \frac{x^3}{3}\Big|_0^n = \frac{n^3}{3}$$

If we subtract the area of the light-shaded rectangles from the sum then the integral becomes an upper bound. The total area of the light-shaded rectangles is n^2 as for any given n the light-shaded rectangles can be stacked on top of each other to reach height x^2. This gives the inequality:

$$1^2 + 2^2 + 3^2 + ... + n^2 - n^2 \leq \int_0^n x^2\, dx = \frac{n^3}{3} \Rightarrow 1^2 + 2^2 + 3^2 + ... + n^2 \leq \frac{n^3}{3} + n^2.$$

(c) We approximate the sum by the integral of the function $x \cdot e^{-x^2}$. The sum equals the area under the first n bars in the following diagram. Apart from the first two bars, the others are so short that they are not visible.



The function xe^{-x^2} is increasing from $x = 0$ up to $x = 1/\sqrt{2}$ and then decreasing when $x > 1/\sqrt{2}$. The sum from the second up to the n-th bar can therefore be upper bounded by the integral of the function from 1 up to n, giving the inequality

$$1 \cdot e^{-1^2} + 2 \cdot e^{-2^2} + \cdots + n \cdot e^{-n^2} \leq 1 \cdot e^{-1^2} + \int_1^n xe^{-x^2}\, dx.$$

The antiderivative of the function xe^{-x^2} is $-\frac{1}{2}e^{-x^2}$, so the integral is at most

$$\int_1^n xe^{-x^2} \leq \int_1^\infty xe^{-x^2} = \tfrac{1}{2} - xe^{-x^2}\big|_1^\infty = \frac{1}{2e}$$

and so the sum is at most $1/e + 1/2e = 3/2e$.

3. Sort the following functions in increasing order of asymptotic growth:

$$2^n, n^n, e^{2^n}, 2^{e^n}, n^{e^2}.$$

(For example, if you are given the functions n^2, n, and 2^n, the sorted list would be $n, n^2, 2^n$.) Show that for every pair of consecutive functions f, g in your list, f is $o(g)$.

Solution: The sorted list is

$$n^{e^2}, 2^n, n^n, e^{2^n}, 2^{e^n}.$$

We now show that for every consecutive pair f, g in the list, $f(n)/g(n) \to 0$ as $n \to \infty$.

(a) $n^{e^2}/2^n = 2^{(\log n)\cdot e^2}/2^n = 2^{e^2 \log n - n} = 2^{-n(1-e^2(\log n)/n)} \to 2^{-\infty(1-0)} = 0$.
(We used the fact that $\log n = o(n)$, so $(\log n)/n \to 0$.)

(b) $2^n/n^n = (2/n)^n \to 0^\infty = 0$.

(c) $n^n/e^{2^n} = e^{(\ln n)\cdot n}/e^{2^n} = e^{n \ln n - 2^n} = e^{-2^n(1-(n \ln n)/2^n)} \to 2^{-\infty(1-0)} = 0$.
(We used the fact that $n \ln n = o(2^n)$, so $(n \ln n)/2^n \to 0$.)

(d) $e^{2^n}/2^{e^n} = 2^{(\log e)\cdot 2^n}/2^{e^n} = 2^{2^n \cdot \log e - e^n} = 2^{-e^n(1-(2/e)^n \log e)} \to 2^{-\infty(1-0)} = 0$.

4. Write each of the following summations S as big-theta of a simple closed-form function f. Prove that S is $O(f)$ and f is $O(S)$.

(a) $n + (n+1) + (n+2) + \cdots + 2n$.

Solution: We can derive an formula for this sum:

$$n + (n+1) + \cdots + 2n = (1 + \cdots + 2n) - (1 + \cdots + (n-1)) = \frac{2n(2n+1)}{2} - \frac{(n-1)n}{2} = \tfrac{3}{2}n^2 + \tfrac{3}{2}n.$$

This is a polynomial in n with leading term n^2 so it is $\Theta(n^2)$.

Alternative solution: S is a sum of $n+1$ terms, each of which is between n and $2n$ so

$$(n+1)\cdot n \le S \le (n+1)\cdot 2n$$

and so S is $\Theta(n(n+1))$, which is the same as $\Theta(n^2)$.

(b) $\log(n) + \log(n+1) + \cdots + \log(2n)$.

Solution: There is no closed-form expression for this sum so we have to resort to approximation. This sum S consists of $n+1$ terms each of which is between $\log n$ and $\log(2n)$ so

$$(n+1)\log n \le S \le (n+1)\log(2n).$$

The left hand side tells us that $n \log n$ is $O(S)$. The right hand side tells us that S is $O(n \log n)$ because

$$(n+1)\log(2n) \le (n+n)\log(n \cdot n) = 2n \log n^2 \le 4n \log n.$$

when $n \ge 1$.

(c) $2^{1^2} + 2^{2^2} + \cdots + 2^{n^2}$. (**Hint:** Use the geometric sum formula.)

Solution: S is at least as large its term 2^{n^2}, and it is at most as large as the sum of *all* powers of two between 0 and n^2, namely

$$2^{n^2} \le S \le 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n^2} = 2 \cdot 2^{n^2} - 1$$

by the geometric sum formula. As S is sandwiched between 2^{n^2} and $2 \cdot 2^{n^2}$ it must be $\Theta(2^{n^2})$.