1. For each of the following functions, say if it is (i) injective (ii) surjective. Justify your answer.

   (a) $f\colon \{0,1\}^3 \to \{0,1,2,3\}$ given by $f((x,y,z)) = x+y+z$.

   Solution: **Not injective** because $f(0,0,1) = 1 = f(1,0,0)$. **Surjective** because evrey value n is obtained by evaluating f on a string with exactly n ones: $0 = f((0,0,0)), 1 = f((1,0,0)), 2 = f((1,1,0)), 3 = f((1,1,1))$.

   (b) $g\colon \{0,1\}^3 \to \{0,1,2,3,4,5,6,7\}$ given by $g((x,y,z)) = x+2y+4z$.

   Solution: **Surjective** because every number is the image of its 3-bit representation zyx. **Injective** because different numbers have distinct bit representations.

   (c) $h\colon \{0,1\}^3 \to \{0,1,3,4,5,6,7,8\}$ given by $h((x,y,z)) = x+3y+4z$.

   Solution: **Not injective** because $f((0,0,1)) = f((1,1,0))$. **Not surjective** because the value 6 is never taken: If $y=0$ or $z=0$ then $h(x,y,z) \le 5$, and if $y=z=1$ then $h(x,y,z) \ge 7$.

2. A password consists of the digits `0` to `9` and the special symbols `*` and `#`. How many 6 to 8-symbol passwords are there if

   (a) the password starts with a `*` and ends with a `#`?

   Solution: The set of passwords P is the disjoint union of the sets of 6, 7, and 8 letter passwords. Each of them is a product set whose first and last symbols are fixed and whose remaining symbols can be any one of the 12 possible symbols S. For example, $P_6 = \{*\} \times S^4 \times \{\#\}$. Therefore

   $$|P| = |P_6| + |P_7| + |P_8| = 12^4 + 12^5 + 12^6 = 3,255,552.$$

   (b) there is at least one special symbol?

   Solution: The set S^6 of all 6-symbol strings is the disjoint union of the set P_6 of 6-symbol passwords and the set D^6 of all 6-digit strings, namely those that do not have a special symbol. By the sum rule $|P_6| = |S^6| - |D^6| = |S|^6 - |D|^6 = 12^6 - 10^6$. By the sum rule again

   $$|P| = |P_6| + |P_7| + |P_8| = (12^6 - 10^6) + (12^7 - 10^7) + (12^8 - 10^8) = 357,799,488.$$

   (c) there is exactly one `*` and exactly two `#`s?

   Solution: We can specify each 6-symbol password by the position of the `*`, the positions of the two `#`, and the sequence of digits in the remaining three positions. By the generalized product rule, $|P_6| = 6 \cdot \binom{5}{2} \cdot 10^3$. By similar reasoning

   $$|P| = |P_6| + |P_7| + |P_8| = 6 \cdot \binom{5}{2} \cdot 10^3 + 7 \cdot \binom{6}{2} \cdot 10^4 + 8 \cdot \binom{7}{2} \cdot 10^5 = 17,910,000.$$

3. How many 8×8 chessboard configurations are there with...

   (a) 4 white rooks, and all must be in different rows and columns?

   Solution: Let us first count the configurations D of four distinct white rooks in different rows and columns. By the generalized product rule, there are 8 choices for the first rooks' row and column, 7 remaining choices for the second rooks' row and column, and so on, giving

   $$|D| = 8 \cdot 8 \cdot 7 \cdot 7 \cdot 6 \cdot 6 \cdot 5 \cdot 5 = (8 \cdot 7 \cdot 6 \cdot 5)^2.$$

   Each configuration of four identical rooks arises out of 4! configurations in which the 4 rooks are labeled. By the division rule the actual number of configurations A is $|A| = (8 \cdot 7 \cdot 6 \cdot 5)^2/4! = 117,600$.

(b) 2 white and 2 black rooks, and all must be in different rows and columns?

Solution: In class we showed that the two white rooks can be placed in $(8 \cdot 7)^2/2$ ways. Once the positions of the white rooks are fixed, by the generalized product rule the black rook configurations are in 1-1 correspondence with the positioning of two black rooks in different rows and columns of a 6×6 board. By the same reasoning these can be placed in $(6 \cdot 5)^2/2$ ways. By the generalized product rule the total number of configurations is $(8 \cdot 7 \cdot 6 \cdot 5)^2/4$.

Alternative solution: If we assign distinct labels to the four rooks the set of configurations is represented by the set D from part (a). Let's label the rooks by their color and identity as $W1$, $W2$, $B1$, $B2$. Forgetting about identities gives a function from D to the set B of unlabeled configurations of 2 white and 2 black rooks of the desired type. Each configuration in B arises from four possible configurations in D so this map is 4-to-1. By the division rule $|B| = |D|/4 = (8 \cdot 7 \cdot 6 \cdot 5)^2/4 = 705,600$.

(c) 2 white and 2 black rooks, all rooks of the same color must be in different rows and columns

Solution: Let us first think of all rooks as distinct and call the set of such configurations D. The white rooks can be placed in $(8 \cdot 7)^2$ ways. Once their positions are fixed, the set of configurations of the (distinct) black rooks is a disjoint union of $B \cup E \cup N$, where B includes those in which the first black rook shares Both a row and a column with a white rook, E includes those in which it shares Either a row or a column with a white rook but not the other, and N includes the remaining configurations.

- The set B provides two choices for placing the first black rook, and 7^2 choices for the second one in distinct rows and columns, so $|B| = 2 \cdot 49 = 98$.
- The set E is a generalized product set: The position of the first black rook can be specified by identifying the shared row or shared column with a white rook (4 choices) and the position within that row of column (6 choices as the positions indexed by the two white rooks cannot be assigned). The second black rook can be placed in any one of $7^2 - 1$ positions in different rows and columns excluding the one that is already occupied by a white rook. Summarizing, $|E| = 4 \cdot 6 \cdot (7^2 - 1) = 1152$.
- The set N is also a generalized product set: The first black rook can be placed in 6^2 ways and the second one in another $7^2 - 2$ ways so $|N| = 6^2 \cdot (7^2 - 2) = 1692$.

We conclude that $|B \cup E \cup N| = |B| + |E| + |N| = 98 + 1152 + 1692 = 2942$ and $|D| = (8 \cdot 7)^2 \cdot 2942$. As each actual configuration is represented by 4 labeled configurations in D, the desired count is $(8 \cdot 7)^2 \cdot 2942/4 = 2,306,528$.

4. Use the pigeonhole principle to prove that

(a) Among any 17 points in the unit square, there is a pair within a distance of at most 0.36.

Solution: Divide the unit square into 16 smaller squares of side length 0.25. Since there are 17 points and only 16 smaller squares, by the pigeonhole principle, at least one smaller square must contain at least two points. The diagonal of each smaller square is $\sqrt{0.25^2 + 0.25^2} = 0.25\sqrt{2}$. Hence, any two points within the same smaller square are at most $0.25\sqrt{2}$ apart, which is less than 0.36.

(b) In every set of 14 numbers between 0 and 42, there are three pairs that have the same sum modulo 43.

Solution: . Let S be the set of numbers and f be the function that takes 2-subsets of S to the sum of their elements modulo 43, namely $f(\{x, y\}) = x + y \bmod 43$. There are $\binom{14}{2} = 91$ possible inputs to f and 43 possible outputs. Since $91 > 2 \cdot 43$ there must be three input pairs that yield the same output.

(c) In every group of at least two people, there are two that have the same number of friends within the group.

Solution: We apply induction on the number n of people in the group. When $n = 2$ either the two people are friends, so both have one friend in the group, or not, so both have zero friends in the group. In either case the base case checks out. Now assume this is true for every group of n people. In a group of $n + 1$ people two cases are possible. If there is a person who has no friends then we can apply the inductive hypothesis to the remaining n people and conclude there must be two with the same number of friends. If not, each one of the $n + 1$ people has between 1 and n friends, so by the pigeonhole principle (applied to the function that maps each person to the number of their friends) two people must have the same number of friends.