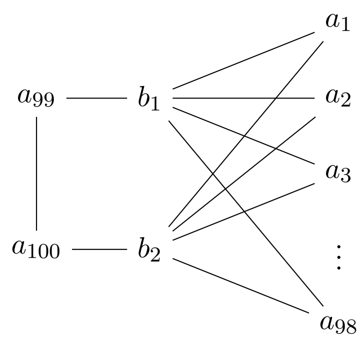
1. Consider the following graph properties and determine if such graphs exist. If they do, provide an example. If not, provide a proof of their non-existence.

   (a) A graph with 100 vertices of degree 3 and 3 vertices of degree 99.

   Solution: No. The sum of the prescribed degrees is $100 \cdot 3 + 3 \cdot 99 = 597$, which is an odd number. This violates Lemma 1 from Lecture 5, which says that the sum of the degrees must equal twice the number of edges, and is therefore always an even number.

   (b) A graph with 100 vertices of degree 2 and 2 vertices of degree 99.

   Solution: Yes. In this graph vertices $a_1, \ldots, a_{100}$ have degree 2 and vertices b_1, b_2 have degree 99.



   (c) A bipartite graph with 100 vertices of degree 2 and 2 vertices of degree 99.

   Solution: No. The sum of the degrees on the two sides of the partition must be the same. First we show that the vertices of degree 99 must be on opposite sides of the partition. If those two vertices were on the same side and all other vertices were on the opposite side then the degree sums on the two sides would be 198 and 200 which are not equal. If both degree-99 vertices and at least one degree-2 vertex were on the same side than that side would have degree sum at least 200 which is strictly larger than the sum of the remaining degrees.

   Therefore the degree-99 vertices must be on opposite sides. For the sum of degrees to be equal there must be 50 degree-2 vertices on one side and 50 on the other side. As each side has 51 vertices there can be no vertex with 99 neighbors.

2. A science fair has participants from schools in X, Y, and Z cities. The table entry $P(r, c)$ in row r and column c represents the average number of projects completed by students from city r in collaboration with students from city c:

	X	Y	Z
X	4	?	2
Y	3	5	1
Z	6	2	3

   (a) Show that $P(r, c)/P(c, r)$ must equal (number of students from c)/(number of students from r).

   Solution: Let's take $r = X$ and $c = Y$. Define S_X as the set of students from city X and S_Y as the set of students from city Y. Consider the collaborative projects between S_X and S_Y. The average number of projects completed by S_X with S_Y is $P(X, Y)$, and the average number of projects completed by S_Y with S_X is $P(Y, X)$. Therefore, the sum of projects between the two sets is $P(X, Y) \cdot |S_X|$ and $P(Y, X) \cdot |S_Y|$, which must be equal: $P(X, Y) \cdot |S_X| = P(Y, X) \cdot |S_Y|$. Hence, $P(X, Y)/P(Y, X) = |S_Y|/|S_X|$.

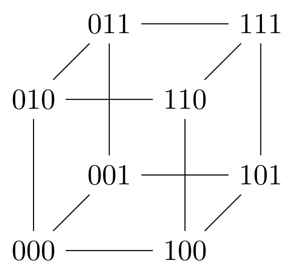
(b) Use part (a) to show that $P(X,Y) \cdot P(Y,Z) \cdot P(Z,X) = P(Y,X) \cdot P(Z,Y) \cdot P(X,Z)$.

Solution:
$$\frac{P(X,Y)}{P(Y,X)} \cdot \frac{P(Y,Z)}{P(Z,Y)} \cdot \frac{P(Z,X)}{P(X,Z)} = \frac{|S_Y|}{|S_X|} \cdot \frac{|S_Z|}{|S_Y|} \cdot \frac{|S_X|}{|S_Z|} = 1.$$

(c) Find the missing entry in the table.

Solution: Applying the identity from part (b), we get $P(X,Y) \cdot 1 \cdot 6 = 3 \cdot 2 \cdot 4$, so $P(X,Y) = 4$. Hence, the missing entry in the table is 4.

3. The *n-dimensional cube* Q_n is a graph on 2^n vertices, where each vertex corresponds to an $\{0,1\}$ string of length n. Two vertices are adjacent if their bit strings differ in exactly one position. Here is a diagram of Q_3:



(a) Show that for every $n \geq 1$, Q_n is a bipartite graph.

Solution: Let E and O be the sets of vertices whose bit string representations contain an even and odd number of ones, respectively. Since every edge in Q_n connects vertices whose bit strings differ in exactly one position, one vertex will have an even number of ones, and the other will have an odd number. Thus, Q_n is bipartite.

(b) Show that for every $n \geq 1$, Q_n has a perfect matching.

Solution: In Q_n, for every vertex represented by a string z of length $n-1$, the edges $\{z0, z1\}$ form a perfect matching because each vertex is included in exactly one edge.

(c) Assuming n is odd, let R_n be the graph obtained by removing all vertices from Q_n except those that have exactly $(n-1)/2$ zeroes or ones. Show that R_n is (i) bipartite and (ii) regular.

Solution: As Q_n is a bipartite graph, its subgraph R_n must also be bipartite: All edges contain one vertex with $(n-1)/2$ zeros and one vertex with $(n-1)/2$ ones. Every vertex has exactly $(n+1)/2$ neighbors, namely those strings obtained by flipping each dominant bit. Therefore the graph must be regular.

(d) By part (d) R_n has a perfect matching for all odd n. Describe perfect matchings for R_3 and R_5.

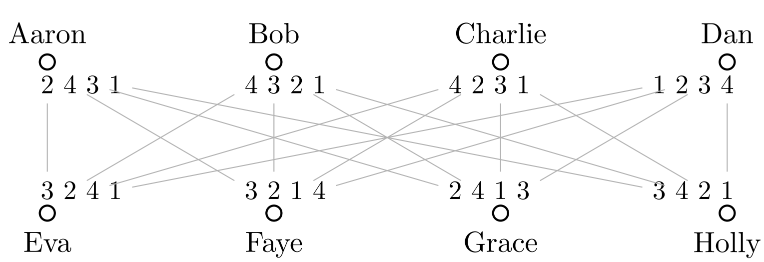
Solution: For G_3, one matching is $\{001, 011\}, \{010, 110\}, \{100, 101\}$. For G_5, one matching is

$$\{00011, 00111\}, \{00110, 01110\}, \{01100, 11100\}, \{11000, 11001\}, \{10001, 10011\}$$

together with

$$\{00101, 01101\}, \{01010, 11010\}, \{10100, 10101\}, \{01001, 01011\}, \{10010, 10110\}.$$

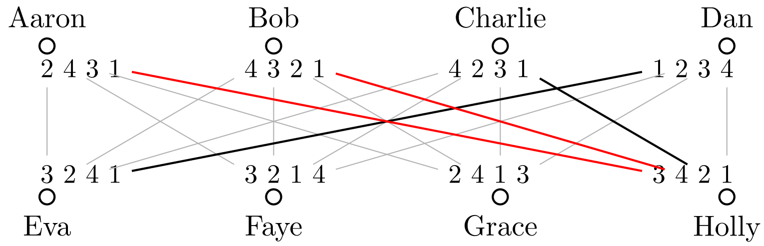
4. Find stable matchings for the following preference lists with (a) boys proposing and girls choosing and (b) girls proposing and boys choosing.



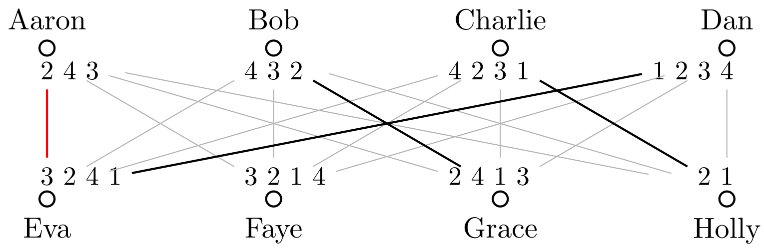
Solution:

(a) The Gale-Shapley algorithm goes through the following rounds:

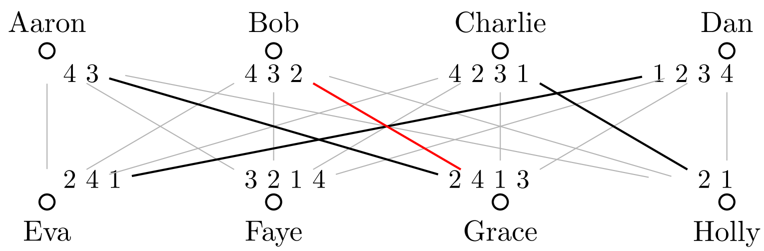
R1: Aaron, Bob, and Charlie propose to Holly. Dan proposes to Eva. Holly rejects Aaron and Bob.



R2: Aaron proposes to Eva and Bob proposes to Grace. Eva rejects Aaron.



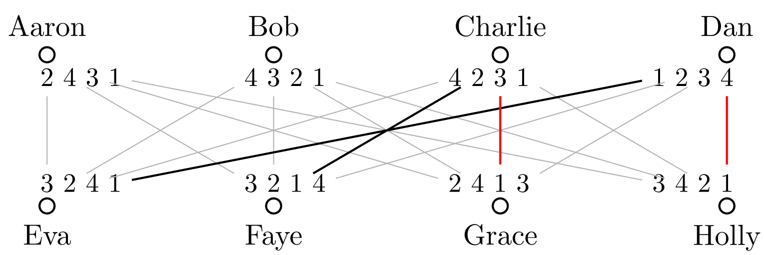
R3: Aaron proposes to Grace. Grace rejects Bob.



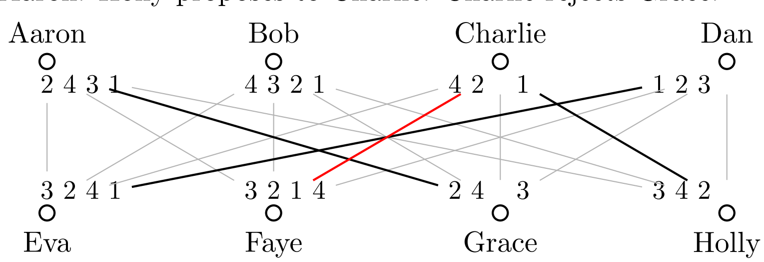
R4: Bob proposes to Faye. No rejections are served. The matching {Aaron, Garce}, {Bob, Faye}, {Charlie, Holly}, {Dan, Eva} is output.

(b) Now the steps are:

R1: Eva and Holly propose to Dan. Faye and Grace propose to Charlie. Charlie rejects Grace. Dan rejects Holly.



R2: Grace proposes to Aaron. Holly proposes to Charlie. Charlie rejects Grace.



R3: Faye proposes to Bob. No further rejections are issued. The same matching as in part (a) is output. As matching (a) is best possible for the boys and matching (b) is worst-possible for the boys this instance does not have any other stable matching.