1. Are the following propositions about graphs true or false? Justify your answer. Specify your proof method.

   (a) Assume G is connected. Let G' be the graph obtained by removing an edge e from G. G' is connected if and only if e belongs to a cycle in G.

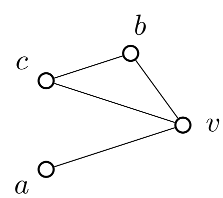
   (b) Assume G is connected. Let G' be the graph obtained by removing a vertex v and its incident edges from G. G' is connected if and only if v belongs to a cycle in G.

   (c) If every vertex in G belongs to a closed walk of odd length then there are at least as many edges as there are vertices in G.

   Solution:

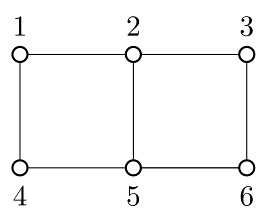
   (a) True. If edge $e = \{u, v\}$ lies on a cycle in G then there is a path p from u to v that does not contain e. Since G is connected, there is a path between every two vertices. If e occurs on this path, it can be replaced by p to obtain a path in G', so the graph remains connected. Conversely, if e does not lie on a cycle then G' cannot be connected because if G' has a path p from u to v then p together with e would form a cycle in G, a contradiction.

   (b) False. This graph below becomes disconnected after removing v:



   (c) True. We first prove this under the assumption that G is connected. If G has an odd-length closed walk then it must have an odd-length cycle. If G has a cycle then it is not a tree. Since G is connected it has more edges than any spanning tree of it. As a tree has one edge less than it has vertices, G must have at least as many edges as it has vertices. In general, applying this reasoning to every connected component of G we conclude that the claim must be true for G.

2. Let G be the graph below. In this question you will count how many spanning trees G has.



   You will make use of the following auxiliary graph H: The vertices of H are the edges of G. A pair $\{e, f\}$ is an edge of H if removing edges e and f from G disconnects it.

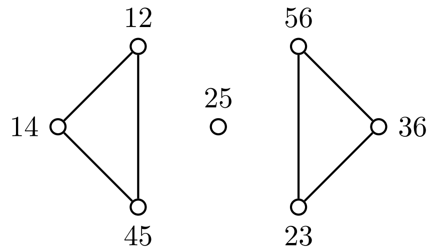
   (a) Draw a diagram of H.

   (b) Argue that the number of spanning trees of G equals the number of vertex-pairs in H that *do not* form an edge.

   (c) Use parts (a) and (b) to count the number of spanning trees of G.

   Solution:

(a) There are two types of "cuts" that disconnect G by removing two edges. One type isolates one of the vertices $1, 3, 4, 6$ from the rest. The other type isolates the pairs $\{1, 4\}$, $\{3, 6\}$ from the rest. Therefore H has 6 edges and looks like this:



(b) Since G has 6 vertices, any spanning tree of it must have 5 edges. Each such spanning tree is obtained by removing a pair of edges that *does not* disconnect the graph. These pairs are represented by pairs of vertices in H that do not form an edge.

(c) By part (b) this equals the number of pairs of vertices in H that do not form an edge. There are $7 \cdot 6/2 = 21$ pairs of vertices in H overall: the first one can be chosen in 7 ways and the second one in 6 remaining ways. This overcounts the pairs by a factor of two as they are unordered. Subtracting the 6 edges from part (a) we obtain a count of 15.

3. In this question you will work out vertex-disjoint paths for the following source-sink pairs in the Beneš network B_3. The sources are labeled `1` to `8` and the sinks are labeled `A` to `H` from top to bottom.

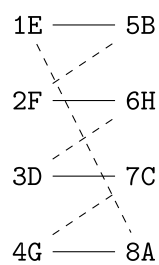
`1E   2F   3D   4G   5B   6H   7C   8A`

(a) For each source-sink pair above, determine whether the path should be routed through the top or through the bottom.

(b) Route the top and bottom paths from part (a) recursively. Draw a diagram of the resulting eight vertex-disjoint paths.

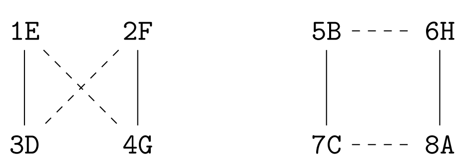
Solution:

(a) The "conflict graph" looks like this, with solid and dashed edges showing conflicts between sources and sinks, respectively:

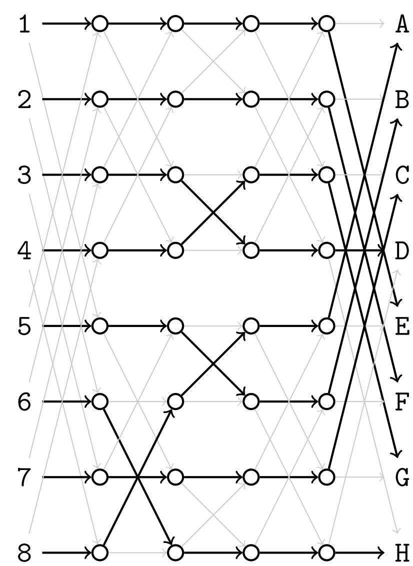


This graph is bipartite with respect to the partition $T = \{\mathtt{1E}, \mathtt{2F}, \mathtt{3D}, \mathtt{4G}\}$ and $B = \{\mathtt{5B}, \mathtt{6H}, \mathtt{7C}, \mathtt{8A}\}$.

(b) The "constraint graphs" for the top and bottom copies of B_2 are



We can route `1E` and `2F` through the top part of the top copy of B_2, `3D` and `4G` through the bottom part, `5B` and `8A` through the top part of the bottom copy of B_2, and `6H` and `7C` through the bottom part of the bottom copy. This yields the following vertex-disjoint paths:



4. Let G be the digraph whose vertices are the 125 3-digit numbers with digits 1, 2, 3, 4, 5, and (u, v) is an edge if $v - u$ equals 1, 10, or 100.

   (a) Show that G is acyclic.

   (b) What is the length of the longest path in G? Justify your answer.

   (c) Use part (b) to show that G must have an antichain of size 10.

   (d) **(Optional)** Show that G has an antichain of size 19.

   (e) **(Optional)** Show that the vertices of G can be partitioned into 19 (vertex-disjoint) paths. Conclude that G cannot have an antichain of size 20.

Solution:

(a) Order the numbers from smallest to largest. This ordering is a topological sort: There can be no back edges because the differences in the backward direction are all negative.

(b) G has a path of length 12:

$$(111, 112, 113, 114, 115, 125, 135, 145, 155, 255, 355, 455, 555).$$

On the other hand, G cannot have a path of length 13 because along every edge at least one of the digits must increase, and this can happen at most four times for each digit.

(c) As G does not have a path of length 13, by Corollary 11 from Lecture 9 it must have an antichain of length at least $125/13 > 9$.

(d) If there is a path from v to w then the sum of digits of w must be greater than the sum of digits of v. Therefore the set of vertices whose digits sum up to 9 is an antichain. There are 19 such vertices.

(e) The following 19 paths are vertex-disjoint and partition all 125 vertices:

$$\begin{aligned} p_a &= (a11, a12, a13, a14, a15, a25, a35, a45, a55), && \text{where } a \in \{1,2,3,4,5\} \\ q_a &= (a21, a22, a23, a24, a34, a44, a54), && \text{where } a \in \{1,2,3,4,5\} \\ r_{bc} &= (1bc, 2bc, 3bc, 4bc, 5bc), && \text{where } b \in \{3,4,5\} \text{ and } c \in \{1,2,3\} \end{aligned}$$

If G had an antichain of size 20, then two of its vertices would have to belong to one of these 19 paths. However, no two vertices in an antichain can belong to the same path.