1. Find and explain the mistakes in the following "proofs".

   (a) **Theorem:** In every group of 5 people there is a person who is friends with at least 3 of them.

   Proof. Let A and B denote two of the five people. The proof is by case analysis. We consider two cases:

   - **Case 1:** A is friends with at least 3 other people in the group.
   - **Case 2:** B is friends with at least 3 other people in the group.

   It follows that at least one of A and B is friends with at least 3 other people, so a person that is friends with at least 3 others always exists. □

   (b) **Theorem:** Every group of 8 people includes a group of 4 friends or a group of 4 strangers.

   Proof. Let A be one of the eight people. The proof is by case analysis. We consider two cases:

   - **Case 1:** A is friends with at least 4 other people in the group.
   - **Case 2:** A is a stranger to at least 4 other people in the group.

   One of these two cases must hold. Let's discuss Case 1. If all the people who are friends with A are strangers among themselves, this is a group of 4 strangers. Otherwise, at least 3 of them are mutual friends, and together with A they form a group of 4 friends.

   Now let's do Case 2. If all the people who are strangers to A are friends among themselves, this is a group of 4 friends. Otherwise, at least 3 of them are mutual strangers, and together with A they form a group of 4 strangers. □

   (c) **Theorem:** In every 3 by 3 table containing the digits `1` to `9` each once, some two consecutive digits must appear in the same row or in the same column.

   Proof. We prove this by contradiction. Once we fix the placement of `1` there are four positions from which `2` is blocked, namely the two in the same column and the two in the same row. When we position `2` in one of the remaining ones, there are now four more positions from which `3` is blocked. We repeat the argument one more time. There are now a total of $3 \times 4 = 12$ blocked positions so `4` cannot be placed anywhere in the table. This is a contradiction so a table with the desired properties cannot exist. □

2. Prove the following theorems using the specified proof method.

   (a) The sum of any three consecutive numbers is a multiple of 3. (Direct implication)

   (b) If a is even or b is even then $a^2 \cdot b$ is even. (Cases)

   (c) If $a^2 \cdot b$ is even then a is even or b is even. (Contrapositive)

   (d) Any 3×3 table containing each of the numbers 1 to 9 exactly once has (at least) two even numbers in the same row. (Contradiction)

3. Prove the following theorems. Specify your proof method.

   (a) For every odd integer n, $3n^2 - 7$ is a multiple of 4.

   (b) For every integer n, the number $n^3 - 3n + 2$ is even.

   (c) For all positive real numbers x and y, if x is irrational, at least one of the numbers $x + y$, $x^2 + y^2$, x^2 is irrational.

4. Which of these propositions are true and which are false? If a proposition is true, prove it. If it is false, prove its negation. (If you claim a number is irrational, provide a proof or give a reference, for example "By Theorem 9 in Lecture 2, $\sqrt{2}$ is irrational.")

   (a) If $-1 \leq x \leq 0$ then $x^3 + 3x^2 + x - 1 < 0$.

   (b) $\sqrt{3} + \sqrt{6}$ is an irrational number.

   (c) For all irrational numbers x, the number $x^2 - \sqrt{2}$ is irrational.

   (d) (**Optional**) $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is an irrational number.