CSI 2101 A/C: Discrete Structures — Homework 3
University of Ottawa, Winter 2026

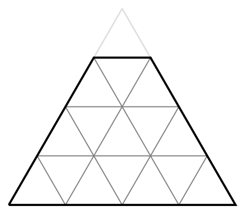
1. Prove the following using induction for every positive integer n.

    (a) The sum of the first n odd integers is n^2.

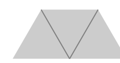
    (b) $1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2 \leq 2 - 1/n$.

    (c) $(2n)!/(n!n!) \leq 4^n$. (**Optional:** $4^n/2n \leq (2n)!/(n!n!)$.)

    (d) The shape C_n is an equilateral triangle of area 4^n is divided into congruent pieces of area 1 with one of the corners removed. C_n can be tiled using the trapezoidal tile T below. *(Source: Daniel J. Velleman. How to Prove it: A structured approach (3rd edition). Exercise 6.2.13)*



Shape C_2



Tile T

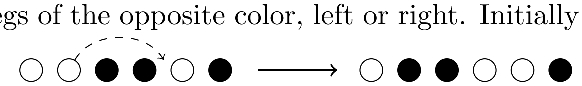
2. Use strong induction to prove the following for all positive integer n.

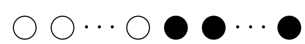
    (a) If $n \geq 14$ then n can be written as $4a + 5b$ for some integers $a, b \geq 0$.

    (b) $(3/2)^{n-2} \leq F_n \leq (7/4)^{n-2}$ for every $n \geq 4$, where F_n is the n-th Fibonacci number ($F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$.)

    (c) $G_n = n!$, where G_n is given by $G_1 = 1$, $G_{n+1} = 1 + G_1 + 2G_2 + \cdots + nG_n$. (**Hint:** $n \cdot n! = (n+1)! - n!$)

3. n white pegs and n black pegs are arranged in a line. In each step you are allowed to move any peg past *two* consecutive pegs of the opposite color, left or right. Initially all white pegs are to the left of the black ones.

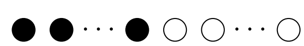




INITIAL

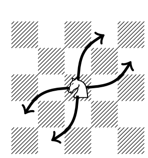
    (a) Assume n is odd. Say a pair of pegs is *inverted* if one is black, one is white, and the black one is to the left of the white one. Prove that "the number of inverted pairs is even" is an invariant.

    (b) If n is odd, can the colors be reversed so that all black pegs are to the left of all white ones?



    (c) If n is even, can the colors be reversed?

4. A knight jumps around an infinite chessboard. Owing to injury it can only make these four moves:



    (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

    (b) Prove that the knight can never reach any one of the four squares adjacent to the initial one by formulating a suitable invariant.

    (c) Can the knight ever end up six squares to the right of its initial position?