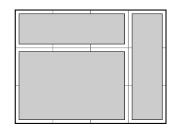
Practice Final 1

1. Are the following propositions true or false? If a proposition is true, prove it. If it is false, prove its negation. Specify your proof method. m and n are integers.
   (a) For every m there exists a n such that $m + 2n = mn$.
   (b) There exists an m such that for every n, $m + 2n = mn$.

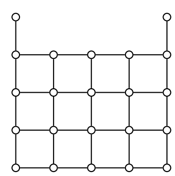
2. You drop 29 balls into 7 urns. Some of the balls are red and some are blue.
   (a) Show that at least three balls of the same color land in the same urn.
   (b) Show that there must be an urn with an unequal number of red and blue balls.

3. Let $f(n)$ be the number of ways to tile a $3 \times n$ field using 1×3 and 2×3 tiles. An example tiling for $n = 4$ is shown on the right.



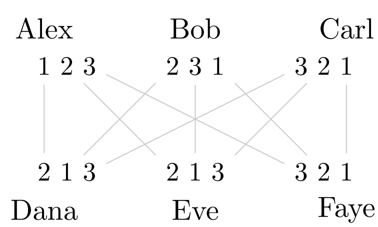
   (a) Fill in the blanks: $f(0) =$ ____ ; $f(1) =$ ____ ; $f(2) =$ ____.
   (b) Write a recurrence for $f(n)$ in terms of $f(n-1)$, $f(n-2)$, and $f(n-3)$. Explain your answer.
   (c) Calculate $f(5)$.

4. Let G be the following graph.



   (a) Show that G is bipartite.
   (b) Does G have a perfect matching? Justify your answer.

5. Find a stable matching for these preferences and show that there is no other stable matching.



Practice Final 2

1. What is the multiplicative inverse of 100 modulo 1009? Show your work.

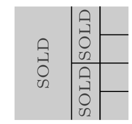
2. A department has 10 men and 15 women. How many ways are there to form a committee with six members if it must have...

   (a) the same number of men and women?

   (b) at least one man and at least one woman?

3. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

4. An $n \times n$ plot of land (n is a power of two) is split in two equal parts by a North-South fence. The Western half is sold and the Eastern half is split in two equal parts by an West-East fence. The same procedure is applied to the remaining $(n/2) \times (n/2)$ plots until 1×1 plots are obtained (see $n = 4$ example).



   (a) Let $T(n)$ be the units of fence used. Write a recurrence for $T(n)$.

   (b) Solve the recurrence.

   (c) Prove that your answer is correct using induction.

5. Let G be the following graph. The vertices of G are all the integers between -10 and 10 except for 0 (20 vertices in total). The pair $\{x, y\}$ is an edge of G if (and only if) $-30 < xy < 0$.

   (a) Show that G is bipartite.

   (b) Show that G does not have a perfect matching.

Practice Final 3

1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".

2. Sort these three functions in increasing order of growth: $\sqrt{n} \cdot \log n$, $n/\sqrt{\log n}$, $\sqrt{n \cdot \log n}$. For your sorted list f, g, h show that f is $o(g)$ and g is $o(h)$.

3. Alice, Bob, and Charlie play a game. Initially Alice holds \$1, Bob holds \$2, and Charlie holds \$5. In each round every player splits their holdings evenly in two and gives them away to the other two players.
   (a) Let $a(n)$ be Alice's holdings after n rounds. Calculate $a(1)$ and $a(2)$.
   (b) Show that $a(n+1) = 4 - a(n)/2$. (**Hint:** The sum of all players' holdings remains invariant.)
   (c) Solve the recurrence from part (b) with initial condition $a(0) = 1$ by unfolding or homogenization.

4. In how many ways can you place 10 white balls and 10 black balls in a 2×10 grid so that there are
   (a) equally many white and black balls in every row?
   (b) equally many white and black balls in every column?
   (c) equally many white and black balls in every row and in every column?

5. G is a directed graph whose vertices are the integers from -10 to 10 (inclusive) and whose edges (x, y) are those ordered pairs for which $|x| - |y| = 1$. For each of the following claims, say if it is true or false and provide a proof.
   (a) G has a path of length 10.
   (b) G has a parallel schedule of duration 11.
   (c) G has an antichain of size 6.

Practice Final 4

1. **Proposition:** Every graph with at least two vertices in which every vertex has odd degree is connected.

   (a) Show that the proposition is false.

   (b) (5 pts) Underline and explain the mistake in the following "proof".

   Proof. We prove the proposition by induction on the number of vertices n. In the base case $n = 2$, G must consist of a single edge. It is connected. For the inductive step, assume that it is true for graphs with $n-1$ vertices. Now suppose G is a graph with n vertices in which every vertex has odd degree. Let v be a vertex of G. Since v has odd degree it has a neighbor w. Remove v from G to obtain a graph G' with $n-1$ vertices, which include w. By the inductive hypothesis, G' is connected. As v has a neighbor in G' (namely w) it is also connected. □

2. Let $a_1 = -1$ and $a_{n+2} = a_n^2 + 2^{n-1}$ for *odd* $n \geq 1$.

   (a) Calculate a_3 and a_5 and fill a number in the blank.

   Theorem: $a_n \equiv$ ___ (mod 3) for all odd $n \geq 1$.

   (b) Prove the Theorem using induction.

3. (a) Calculate the *gcd* of 82 and 18 using Euclid's algorithm. Show your work.

   (b) Let a and b be the results of dividing 82 and 18 by their *gcd*, respectively. Express 1 as an integer linear combination of a and b.

4. Let $f(n)$ be the number of ways to arrange n balls from left to right in which balls are Blue, Green, or Red, and no consecutive Red balls are allowed. For example, when $n = 3$ the arrangements BBG and BRB are counted, but BRR is not.

   (a) Calculate $f(1)$, $f(2)$, and $f(3)$.

   (b) Fill in the blanks in the recurrence. Justify your answer.

   $$f(n) = \_\_\_ \cdot f(n-1) + \_\_\_ \cdot f(n-2).$$

   (c) Calculate the number a for which $f(n)$ is $\Theta(a^n)$.

5. Let G_n the directed graph whose vertices are the integers $\{2, 3, \ldots, n\}$ and whose edges are those pairs $i \to j$ for which $i < j$ and i divides j.

   (a) Calculate the out-degrees and in-degrees of all the vertices in G_8. (**Hint:** Draw G_8 first.)

   (b) Show that (for every n and i) the outdegree of vertex i in G_n is between $n/i - 2$ and n/i.

   (c) Use part (b) to show that G_n has $O(n \log n)$ edges.