Practice Final 1

1. Are the following propositions true or false? If a proposition is true, prove it. If it is false, prove its negation. Specify your proof method. m and n are integers.

   (a) For every m there exists a n such that $m + 2n = mn$.

   Solution: False. We show a counterexample. If $m = 5$ then n must satisfy $5 + 2n = 5n$, so n must equal $3/5$ which is not an integer.

   (b) There exists an m such that for every n, $m + 2n = mn$.

   Solution: False. For contradiction suppose such an m existed. Then $m(1 - n) + 2n = 0$ for all n. But when $n = 1$, the left hand side equals 2 and the right hand side equals zero so $2 = 0$, a contradiction.

2. You drop 29 balls into 7 urns. Some of the balls are red and some are blue.

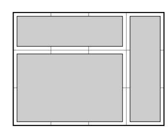
   (a) Show that at least three balls of the same color land in the same urn.

   Solution: Let $f\colon \{1, \ldots, 29\} \to \{1, \ldots 7\} \times \{\text{red}, \text{blue}\}$ be the function that assigns each ball to its urn and its colour. The domain of f has size 29 and its range has size $7 \cdot 2 = 14$. Since $29 > 2 \cdot 14$, by the generalized pigeonhole principle there exist 3 balls that are assigned to the same urn and have the same colour.

   (b) Show that there must be an urn with an unequal number of red and blue balls.

   Solution: We argue by contrapositive. If every urn has an equal number of red and blue balls then it has an even number of balls. The the total number of balls must then also be even. So there cannot be 29 balls.

3. Let $f(n)$ be the number of ways to tile a $3 \times n$ field using 1×3 and 2×3 tiles. An example tiling for $n = 4$ is shown on the right.



   (a) Fill in the blanks:

   Solution:

   $$f(0) = \underline{1} \qquad f(1) = \underline{1} \qquad f(2) = \underline{2}$$

   When $n = 0$, the empty tiling is the only possible tiling. When $n = 1$, there is one tiling with a vertical 3×1 tile. When $n = 2$, there are two tilings: One with two vertical 3×1 tiles and one with a single 3×2 tile.

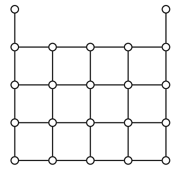
   (b) Write a recurrence for $f(n)$ in terms of $f(n - 1)$, $f(n - 2)$, and $f(n - 3)$. Explain your answer.

   Solution: $f(n) = f(n - 1) + f(n - 2) + 3f(n - 3)$. The set of tilings is a disjoint union of those that start with a 3×1 vertical tile, those that start with a 3×2 vertical tiles, and those that fill up the first three columns with horizontal tiles. There are $f(n - 1)$, $f(n - 2)$, and $3f(n - 3)$ of each type, respectively. By the sum rule we obtain the above recurrence.

   (c) Calculate $f(5)$.

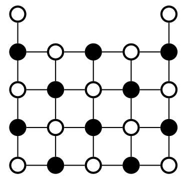
   Solution: We iterate the recurrence to obtain $f(3) = 2 + 1 + 3 \cdot 1 = 6$, $f(4) = 6 + 2 + 3 \cdot 1 = 11$, and $f(5) = 11 + 6 + 3 \cdot 2 = 23$.

4. Let G be the following graph.



(a) Show that G is bipartite.

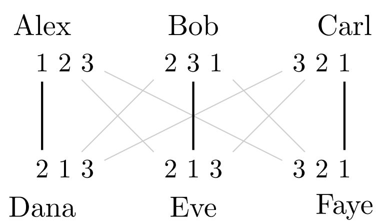
Solution: The following partition of the vertices into White and Black demonstrates that G is bipartite as there are no edges within W or within B.



(b) Does G have a perfect matching? Justify your answer.

Solution: No. There are 10 vertices in B and 12 in W, so the vertices in W cannot all be matched.

5. Find a stable matching for these preferences and show that there is no other stable matching.



Solution: Consider the marked matching $\{\text{Alex}, \text{Dana}\}, \{\text{Bob}, \text{Eve}\}, \{\text{Carl}, \text{Faye}\}$. We show that no other matching is stable. As a stable matching always exists, this one must be stable.

In any stable matching, Carl must be matched to Faye because they are each other's first choice (so they would be a rogue couple if not matched). The same holds for Bob and Eve. Finally, Alex and Dana have to be matched, regardless of their preferences, since there is no other choice left. This leaves the above matching as the only stable possibility, as any change to it would create a rogue couple, hence making it unstable.

Alternative solution: If we run the Gale-Shapley algorithm, on day 1 Alex proposes to Dana, Bob proposes to Eve, and Carl proposes to Faye. Eve picks Bob, Faye picks Carl and Dana picks Alex. The final matching is $\{\text{Alex}, \text{Dana}\}, \{\text{Bob}, \text{Eve}\}, \{\text{Carl}, \text{Faye}\}$. We proved in Lecture 5 that this is stable.

Let us now run the Gale-Shapley algorithm again, but with the girls doing the proposing this time around. On day 1 Dana and Eve propose to Bob and Faye proposes to Carl. Carl picks Faye and Bob picks Eve over Dana. On day 2 Dana proposes to Alex resulting in the same final stable matching.

By Theorem 6 in Lecture 8, the first matching is the best possible for the boys (every boy gets his best possible choice among all stable matchings), while the second one is the worst possible for the boys (every boy gets his worst possible choice). Since they are the same there can be only one stable matching.

Practice Final 2

1. What is the multiplicative inverse of 100 modulo 1009? Show your work.

Solution: We look for numbers s and t such that $100s + 1009t = 1$. The extended Euclid's algorithm goes through the steps

$$1009 = 10 \cdot 100 + 9$$
$$100 = 11 \cdot 9 + 1,$$

from where

$$1 = 100 - 11 \cdot 9 = 100 - 11 \cdot (1009 - 10 \cdot 100) = 111 \cdot 100 - 11 \cdot 1009$$

so we can set $s = 111$ and $t = -11$. Therefore $100 \cdot 111 \equiv 1 \mod 1009$ and 111 is the desired multiplicative inverse.

2. A department has 10 men and 15 women. How many ways are there to form a committee with six members if it must have...

   (a) the same number of men and women?

   Solution: We apply the product rule. There are $\binom{10}{3}$ ways to choose the three men in the committee and $\binom{15}{3}$ ways to choose the three women in it, so the number of possible committees is $\binom{10}{3} \cdot \binom{15}{3}$.

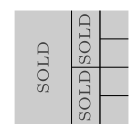
   (b) at least one man and at least one woman?

   Solution: The number of possible committees is $\binom{25}{6}$, the number of men-only committees is $\binom{10}{6}$, and the number of women-only committees is $\binom{15}{6}$. By the sum rule the number is $\binom{25}{6} - \binom{15}{6} - \binom{10}{6}$.

3. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

   Solution: We can prove this proposition by cases depending on the residue of n^3+n modulo 3. If $n \equiv 0 \mod 3$ then $n^3 + n$ is divisible by 3, while $2n^3 + 1 \equiv 1 \mod 3$, so $2n^3 + 1$ is not divisible by 3, so the proposition holds. If $n \equiv 1 \mod 3$ then $n^3 + n \equiv 2 \mod 3$, so $n^3 + n$ is not divisible by 3 and the proposition holds again. If $n \equiv 2 \mod 3$, then $n^3 + n \equiv 10 \equiv 1 \mod 3$ and $n^3 + n$ is not divisible by 3 again.

4. An $n \times n$ plot of land (n is a power of two) is split in two equal parts by a North-South fence. The Western half is sold and the Eastern half is split in two equal parts by an West-East fence. The same procedure is applied to the remaining $(n/2) \times (n/2)$ plots until 1×1 plots are obtained (see $n = 4$ example). How many units of fence are used?



   (a) Let $T(n)$ be the units of fence used. Write a recurrence for $T(n)$.

   Solution: The amount of fence for an $n \times n$ field is twice the amount used for a field half the size plus n units of vertical fence plus $n/2$ units of horizontal fence, giving the recurrence $T(n) = 2T(n/2) + 3n/2$ for $n > 1$. The initial condition is $T(1) = 0$.

   (b) Solve the recurrence.

   Solution: We can unwind the recurrence as follows:

$$\begin{aligned} T(n) &= 2T(n/2) + 3/2 \cdot n \\ &= 2(2T(n/2^2) + 3/2 \cdot n/2) + 3/2 \cdot n = 2^2 T(n/2^2) + 3/2 \cdot 2n \\ &= 2^2(2T(n/2^3) + 3/2 \cdot n/2^2) + 3/2 \cdot 2n = 2^3 T(n/2^3) + 3/2 \cdot 3n \end{aligned}$$

   After $\log n$ steps we expect to obtain $T(n) = n \cdot T(1) + \frac{3}{2} n \log n = \frac{3}{2} n \log n$.

   (c) Prove that your answer is correct using induction.

   For the base case $n = 1$, $T(1) = 0$ as desired. For the inductive step we assume $T(k) = \frac{3}{2} k \log k$ for all $k < n$ that are powers of two. Then

$$T(n) = 2T(n/2) + 3n/2 = 2 \cdot \frac{3}{2} \cdot \frac{n}{2} \log(n/2) + \frac{3n}{2} = \frac{3n}{2} \cdot (\log n - 1) + \frac{3n}{2} = \frac{3}{2} \cdot n \log n$$

   when n is a power of two, concluding the inductive step.

5. Let G be the following graph. The vertices of G are all the integers between -10 and 10 except for 0 (20 vertices in total). The pair $\{x, y\}$ is an edge of G if (and only if) $-30 < xy < 0$.

   (a) Show that G is bipartite.

   Solution: G is bipartite with respect to the partition N, P with $N = \{-10, \ldots, -1\}$ and $P = \{1, \ldots, 10\}$: If x and y are both in P or both in N then $xy > 0$ so $\{x, y\}$ is not an edge of G.

(b) Show that G does not have a perfect matching.

Solution: To show that G has no perfect matching, we exhibit a subset S of P of size 5 whose neighbour set has size at most 4. By Hall's theorem, the vertices in S cannot all be matched. Take $S = \{6, 7, 8, 9, 10\}$. The vertices -5, -6, -7, -8, -9 and -10 are not neighbours of S since any product between one of this numbers and a number in S is at most $-5 \cdot 6 = -30$. Therefore S can have at most 4 neighbours.

Practice Final 3

1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".

   Solution: $\forall u, b_1, b_2\colon IN(b_1, u)$ AND $IN(b_2, u) \longrightarrow b_1 = b_2$. Any two balls in any given urn must be the same ball.

2. Sort these three functions in increasing order of growth: $\sqrt{n} \cdot \log n$, $n/\sqrt{\log n}$, $\sqrt{n \cdot \log n}$. For your sorted list f, g, h show that f is $o(g)$ and g is $o(h)$.

   Solution: $\sqrt{n \log n}$ is $o(\sqrt{n} \log n)$ because the ratio $\sqrt{n \log n}/\sqrt{n} \log n$ equals $1/\sqrt{\log n}$, which eventually becomes and stays smaller than any given constant. $\sqrt{n} \log n$ is $o(n/\sqrt{\log n})$ because the ratio $\sqrt{n} \log n/(n/\sqrt{\log n})$ equals $(\log n)^{3/2}/n^{1/2}$. In Lecture 7 we showed that $(\log n)^a$ is $o(n^b)$ for any constants $a, b > 0$, so this ratio becomes and stays smaller than any constant when n is sufficiently large.

3. Alice, Bob, and Charlie play a game. Initially Alice holds \$1, Bob holds \$2, and Charlie holds \$5. In each round every player splits their holdings evenly in two and gives them away to the other two players.

   (a) Let $a(n)$ be Alice's holdings after n rounds. Calculate $a(1)$ and $a(2)$.

   Solution: $a(1) = (2 + 5)/2 = 7/2$. To calculate $a(2)$ we need to know how much Bob and Charlie hold in round 1: Bob holds $b(1) = (1 + 5)/2 = 3$ and Charlie holds $c(1) = (1 + 2)/2 = 3/2$. Then $a(2) = (3 + 3/2)/2 = 9/4$.

   (b) Show that $a(n + 1) = 4 - a(n)/2$. (**Hint:** The sum of all players' holdings remains invariant.)

   Solution: For every n, $a(n) + b(n) + c(n)$ must equal 8. Therefore $a(n + 1) = (b(n) + c(n))/2 = (8 - a(n))/2 = 4 - a(n)/2$.

   (c) Solve the recurrence from part (b) with initial condition $a(0) = 1$ by unfolding or homogenization.

   Solution: We unwind the recurrence:

$$\begin{aligned}
a(n) &= -\frac{1}{2}a(n-1) + 4 \\
&= -\frac{1}{2}\Big(-\frac{1}{2}a(n-2) + 4\Big) + 4 = \Big(-\frac{1}{2}\Big)^2 a(n-2) - \frac{1}{2} \cdot 4 + 4 \\
&= \Big(-\frac{1}{2}\Big)^2 \Big(-\frac{1}{2}a(n-3) + 4\Big) - \frac{1}{2} \cdot 4 + 4 = \Big(-\frac{1}{2}\Big)^3 a(n-3) + \Big(-\frac{1}{2}\Big)^2 \cdot 4 - \frac{1}{2} \cdot 4 + 4 \\
&= \cdots \\
&= \Big(-\frac{1}{2}\Big)^n a(0) + \Big(-\frac{1}{2}\Big)^{n-1} \cdot 4 + \Big(-\frac{1}{2}\Big)^{n-2} \cdot 4 + \cdots + 4.
\end{aligned}$$

   Plugging in $a(0) = 1$ and using the geometric sum formula we get

$$a(n) = \Big(-\frac{1}{2}\Big)^n + 4 \cdot \frac{1 - (-1/2)^n}{3/2} = \frac{8}{3} - \frac{5}{3} \cdot \Big(-\frac{1}{2}\Big)^n.$$

   This evaluates to $a(0) = 1, a(1) = 7/3, a(2) = 9/4$ which is consistent with part (a).

Alternative solution: We try the homogenization $a(n) = a'(n) + c$ to get $a'(n+1) + c = 4 - (a'(n) + c))/2 = 4 - a'(n)/2 - c/2$, from where c must equal $8/3$. We solve $a'(n)$ to

$$a'(n) = (-1/2)a'(n-1) = (-1/2)^2 a'(n-2) = \cdots = (-1/2)^n a'(0) = (-1/2)^n \cdot (-5/3).$$

Therefore $a(n) = 8/3 - (5/3)(-1/2)^n$.

4. In how many ways can you place 10 white balls and 10 black balls in a 2×10 grid so that there are

(a) equally many white and black balls in every row?

Solution: Each row must contain five white balls and five black balls. There are $\binom{10}{5}$ ways to arrange the balls in the first row (the arrangement can be viewed as a string in $\{W, B\}^{10}$ with five Ws). There are as many in the second row. By the product rule the total number is $\binom{10}{5}^2$.

(b) equally many white and black balls in every column?

Solution: There are 2^{10} possible arrangements of the first row (the arrangement can be any string in $\{W, B\}^{10}$). Once the first row is fixed, there is exactly one possible arrangement of the second row obtained by swapping the color in each column. By the generalized product rule the total number is 2^{10}.

(c) equally many white and black balls in every row and in every column?

Solution: There are not $\binom{10}{5}$ possible arrangements of the first row as there must be exactly five white balls. Once the first row is fixed, there is again exactly one possible arrangement of the second row. By the generalized product rule the number is $\binom{10}{5}$.

5. G is a directed graph whose vertices are the integers from -10 to 10 (inclusive) and whose edges (x, y) are those ordered pairs for which $|x| - |y| = 1$. For each of the following claims, say if it is true or false and provide a proof.

(a) G has a path of length 10.

Solution: True. The path $(10, 9, 8, \ldots, 0)$ has length 10.

(b) G has a parallel schedule of duration 11.

Solution: True. G cannot have a path of length 11 because the absolute value of vertices starts at 10 and must decrease along any path. As G has a parallel schedule whose length is the maximum path length plus one it must have a parallel schedule of size 11.

(c) G has an antichain of size 6.

Solution: False. There cannot be an antichain of size 3 or larger because among any 3 numbers there must be a pair x, y that are different in absolute value. Therefore there is a path from the larger to the smaller so x, y cannot be in an antichain.

Practice Final 4

1. **Proposition:** Every graph with at least two vertices in which every vertex has odd degree is connected.

(a) Show that the proposition is false.

Solution: In a perfect matching on four vertices all vertices have degree 1, which is odd, but the graph is not connected.

(b) Underline and explain the mistake in the following "proof".

Proof. We prove the proposition by induction on the number of vertices n. In the base case $n = 2$, G must consist of a single edge. It is connected. For the inductive step, assume that it is true for graphs with $n-1$ vertices. Now suppose G is a graph with n vertices in which every vertex has odd degree. Let v be a vertex of G. Since v has odd degree it has a neighbor w. Remove v from G to obtain a graph G' with $n-1$ vertices, which include w. By the inductive hypothesis, G' is connected. As v has a neighbor in G' (namely w) it is also connected. □

Solution: The inductive hypothesis may not apply to G'. For example in the graph above, after removing v vertex w in has degree zero in G', which is even.

2. Let $a_1 = -1$ and $a_{n+2} = a_n^2 + 2^{n-1}$ for *odd* $n \geq 1$.

   (a) Calculate a_3 and a_5 and fill a number in the blank.

   Theorem: $a_n \equiv \underline{2} \pmod 3$ for all odd $n \geq 1$.

   Solution: $a_3 = (-1)^2 + 2^0 = 2$ and $a_5 = 2^2 + 2^2 = 8$. Both are 2 modulo 3 as is a_1.

   (b) Prove the Theorem using induction.

   Solution: Base case: $a_1 = -1 \equiv 2 \pmod 3$.
   Inductive hypothesis: $a_n \equiv 2 \pmod 3$ where n is odd.
   Inductive step: As n is odd it equals $2k+1$ for some k. Then

$$a_{n+2} = a_n^2 + 2^{n-1} \equiv 2^2 + 2^{2k+1-1} = 4 + 4^k = 1 + 1^k \equiv 2 \pmod 3.$$

3. (a) Calculate the *gcd* of 82 and 18 using Euclid's algorithm. Show your work.

   Solution:

$$\begin{aligned} E(82,18) &= E(18,10) & \qquad 82 &= 4\cdot 18 + 10 \\ &= E(10,8) & 18 &= 10 + 8 \\ &= E(8,2) & 10 &= 8 + 2 \\ &= E(2,0) & 8 &= 4 \cdot 2 + 0 \\ &= 2. \end{aligned}$$

   (b) Let a and b be the results of dividing 82 and 18 by their *gcd*, respectively. Express 1 as an integer linear combination of a and b.

   Solution: We first apply extended Euclid's algorithm to express 2 as a combination of 82 and 18:

$$2 = 10 - 8 = 10 - (18 - 10) = -18 + 2 \cdot 10 = -18 + 2 \cdot (82 - 4 \cdot 18) = 2 \cdot 82 - 9 \cdot 18.$$

   Dividing both sides by 2 we get

$$1 = 2 \cdot (82/2) - 9 \cdot (18/2) = 2a - 9b.$$

4. Let $f(n)$ be the number of ways to arrange n balls from left to right in which balls are <u>B</u>lue, <u>G</u>reen, or <u>R</u>ed, and no consecutive <u>R</u>ed balls are allowed. For example, when $n = 3$ the arrangements <u>BBG</u> and <u>BRB</u> are counted, but <u>BRR</u> is not.

   (a) Calculate $f(1)$, $f(2)$, and $f(3)$.

$$f(1) = \underline{3} \qquad f(2) = \underline{8} \qquad f(3) = \underline{22}.$$

   When n is 1 either of the three balls forms a valid arrangement. When $n = 2$ there are 3 choices for the first ball and 3 choices for the second one, but the arrangement <u>RR</u> is excluded resulting in $3^2 - 1 = 8$ arrangements. Out of these 8, 2 end in a red ball and the other 6 don't. When $n = 3$, the 2 arrangements that end in a red ball can be arbitrarily extended with a third ball, but the other two can only be extended with a blue or a green ball, giving a total of $6 \cdot 3 + 2 \cdot 2 = 22$.

(b) Fill in the blanks in the recurrence. Justify your answer.

$$f(n) = \underline{2} \cdot f(n-1) + \underline{2} \cdot f(n-2).$$

Solution: The arrangements of n balls split into a disjoint union of four types: Those that start with a blue ball, those that start with a green ball, those that start with a red and a blue ball, and those that start with a red and a green ball. There are $f(n-1)$ arrangements of the first two types and $f(n-2)$ of the second two. The reason is that the remaining balls can be configured into any valid arrangement of length $n-1$ and $n-2$ respectively.

(c) Calculate the number a for which $f(n)$ is $\Theta(a^n)$.

Solution: The solutions of the recurrence are combinations of x_1^n and x_2^n, where x_1 and x_2 are the roots of $x^2 - 2x - 2$, namely $x_1 = 1 + \sqrt{3}$ and $x_2 = 1 - \sqrt{2}$. As $1 - \sqrt{3}$ is less than zero but $f(n)$ must grow with n, x_1^n must appear in the solution for $f(n)$ so $f(n)$ is $\Theta((1 + \sqrt{3})^n)$.

5. Let G_n the directed graph whose vertices are the integers $\{2, 3, \ldots, n\}$ and whose edges are those pairs $i \to j$ for which $i < j$ and i divides j.

(a) Calculate the out-degrees and in-degrees of all the vertices in G_8. (**Hint:** Draw G_8 first.)

Solution:

vertex i	2	3	4	5	6	7	8
out-degree of i	3	1	1	0	0	0	0
in-degree of i	0	0	1	0	2	0	2

(b) Show that (for every n and i) the outdegree of vertex i in G_n is between $n/i - 2$ and n/i.

Solution: There are outgoing edges from i to $2i$, $3i$, and so on, up to the ki for the largest k such that $ki \leq n$, so $k-1$ in total. As ki does not exceed n, k is at most n/i, and so is $k-1$. As $(k+1)i > n$, k is greater than $n/i - 1$, so $k-1$ is greater than $n/i - 2$.

(c) Use part (b) to show that G_n has $O(n \log n)$ edges.

Solution: The number of edges equals the sum of the out-degrees, which is at most

$$\frac{n}{2} + \frac{n}{3} \cdots + \frac{n}{n} = n\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \leq n \ln(n+1).$$

As $\ln(n+1)$ is $O(\log n)$ the sum is $O(n \log n)$.