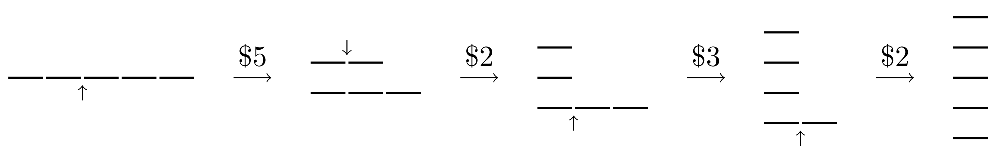
Practice Midterm 1

1. A 3 by 3 table T is filled with the numbers 1 to 9 in arbitrary order, each number occurring exactly once. True or false? Justify your answer. Specify your proof method.
   (a) There exists a table T in which the sum of every column is even.
   (b) There exists a table T in which the sum of every column is odd.

2. A bug sits at the origin 0. In each step it can jump by 21 positions to the left or by 37 positions to the right.
   (a) Show that the bug can reach all integer positions. (**Hint:** Start with position 1.)
   (b) How many left jumps and how many right jumps should the bug take to reach position 1?

3. A stick of length n is to be broken up into n unit sticks in a sequence of moves. A move consists of splitting a long stick into two shorter sticks. The cost of the move in dollars is the length of the stick being split.

   For example, here is a sequence of moves that breaks up a stick of length 5 at a cost of 12 dollars:



   (a) Use strong induction to prove: For every $n \geq 1$, the total cost of breaking up a stick of length n can be at most $\frac{1}{2}n^2 + \frac{1}{2}n - 1$ dollars, regardless of the sequence of moves.
   (b) Show that there exists a move sequence of cost $\frac{1}{2}n^2 + \frac{1}{2}n - 1$ dollars that breaks up a length n stick.

4. In "baby RSA" with modulus $p = 17$, Eve intercepted the ciphertext $c = 4$. The public key is $e = 3$.
   (a) What is Bob's private key d?
   (b) What is Alice's message m?

Practice Midterm 2

1. Express the sentence "Any two people who are not friends have a friend in common" using quantifiers and logical operators. Use x, y, z as variables and $F(x, y)$ for "x and y are friends."

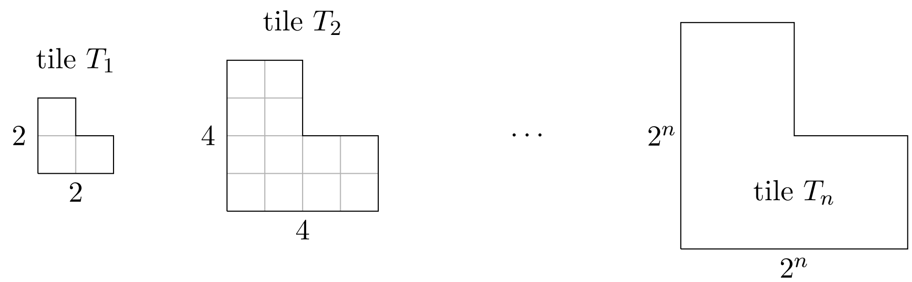
2. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

3. True or false? Justify your answer. Specify your proof method.

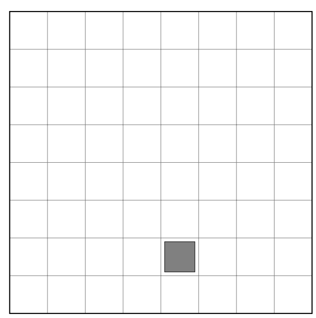
   (a) For all integers a, b, c, at least one of the three numbers $a + b$, $b + c$, $c + a$ is even.

   (b) For all integers a, b, c, at least one of the three numbers $a + b$, $b + c$, $c + a$ is odd.

4. **Claim:** For every $n \geq 1$, a $2^n \times 2^n$ board with one square removed (in any position) can be filled with tiles $T_1, \ldots, T_n$ below (one of each type).



   (a) Describe a tiling for the following board ($n = 3$ with square $(5, 2)$ missing).



   (b) Prove the claim. Specify your proof method.

Practice Midterm 3

1. Underline and explain the mistake in the following "proof."

   Proposition. In every group of friends there exists a person with an even number of friends.

   Proof. By induction on the number of people n. When $n = 1$ the one person has zero friends, and zero is even. Now assume it is true for groups of n people. Let G be a group of $n + 1$ people. Take out any person from G. By inductive hypothesis the remaining group G' has someone, say Alice, with an even number of friends. Since Alice is also in G, G has a person with an even number of friends. $\square$

2. For which nonzero integers n is the number $\frac{\sqrt{2}}{n} - \frac{n}{\sqrt{2}}$ rational? Justify your answer.

3. Alice has infinitely many \$6, \$10, and \$15 stamps.

   (a) Can she make all integer postages of \$30 and above?

   (b) What if the \$10 stamp was replaced by a \$9 stamp?

4. Bob has 32 *b*lue, 33 *r*ed, and 34 *g*reen balls. At every turn he takes out two balls and replaces them with two different balls by the following replacement rule:

   $$bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb.$$

   (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

   (b) Can Bob obtain 99 balls of the same color? Justify your answer.
   (**Hint:** Look at the difference between the number of red and blue balls.)

Practice Midterm 4

1. Are the propositions "Every two people have a common friend" and "Every person has at least two friends" logically equivalent? Justify your answer.

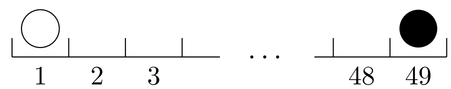
2. Show that for every positive real number x, at least one of the numbers $\sqrt{x}+1$ and $\sqrt{2}\cdot x$ is irrational.

3. Bob is waiting for a secret message from Alice. He publishes RSA modulus $n = 21$ and public key $e = 5$.

   (a) Alice's message is $m = 8$. What is the ciphertext that she sends out? Show your calculations.

   (b) Alice sends another ciphertext $c = 2$ and this one is intercepted by Eve. What was Alice's message?

4. A long ledge is divided into slots numbered from 1 to 49.



   A white ball and a black ball are placed in the first and last slot, respectively. In every step one of the balls is moved 5 slots to the left *or* 10 slots to the right of its current position.

   (a) Formulate this process as a state machine. Describe the states, start state, and transitions.

   (b) *Invariant:* The slot numbers of the black and white balls differ by ____ modulo ____ .
   Fill in the blanks and provide a proof of invariance.

   (c) Can the two balls ever occupy adjacent slots? Justify your answer.