1. Prove the following using induction. $n$ is a positive integer.

   (a) For every $n$, the sum of the first $n$ odd integers is $n^2$.

   **Solution:** Let $S(n)$ be the sum of the first $n$ odd positive integers. We prove that $S(n) = n^2$ for all $n \geq 1$ by induction on $n$.
   **Base case $n = 1$:** $S(1) = 1$ and $1^2 = 1$, so the base case holds.
   **Inductive step:** We assume that $S(n)$ equals $n^2$. The $(n+1)$-st odd positive integer is $2n+1$ and so

   $$S(n+1) = S(n) + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

   It follows by induction that $S(n) = n^2$ for all positive integers $n$.

   (b) For every $n$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + .... + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

   **Solution: Base case $n = 1$:** $1/1^2 = 1 = 2 - 1/1$.
   **Inductive step:** We assume that $1/1^2 + \cdots + 1/n^2 \leq 2 - 1/n$. By the inductive hypothesis

   $$\frac{1}{1^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \left(\frac{1}{n} - \frac{1}{(n+1)^2}\right).$$

   We will now show that the expression in the parenthesis is *at least* $1/(n+1)^2$, so that the right hand side is at most $2 - 1/(n+1)^2$ completing the inductive step.

   $$\frac{1}{n} - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - n}{n(n+1)^2} = \frac{n^2+n+1}{n(n+1)^2} > \frac{n^2+n}{n(n+1)^2} = \frac{1}{n+1}.$$

   (c) $(2n)!/(n!n!) \leq 4^n$. (**Optional:** $4^n/2n \leq (2n)!/(n!n!)$.)

   **Solution: Base Case:** When $n = 1$, $4^1/2 = 2$, $2!/(1!1!) = 2$, and $4^1 = 4$ so both inequalities hold.
   **Inductive Step:** We assume that

   $$\frac{(2n)!}{n!n!} \leq 4^n$$

   and deduce that

   $$\frac{(2(n+1))!}{(n+1)!(n+1)!} = \frac{(2n)! \cdot (2n+1)(2n+2)}{n! \cdot (n+1) \cdot n! \cdot (n+1)} = \frac{(2n)!}{n!n!} \cdot \frac{2n+1}{n+1} \cdot \frac{2n+2}{n+1} \leq 4^n \cdot 2 \cdot 2 \leq 4^{n+1}$$

   completing the inductive step.
   For the optional part, the base case is $4^1/2 = 2 = 2!/(1!1!)$. For the inductive step we assume

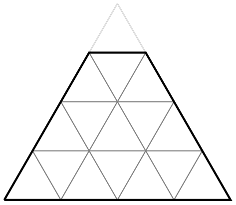
   $$\frac{4^n}{2n} \leq \frac{(2n)!}{n!n!}$$

   and deduce that

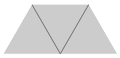
   $$\frac{4^{n+1}}{2(n+1)} = \frac{4^n}{2n} \cdot \frac{4n}{n+1} \leq \frac{(2n)!}{n!n!} \cdot \frac{4n}{n+1} = \frac{(2(n+1))!}{(n+1)!(n+1)!} \cdot \frac{(n+1)^2}{(2n+1)(2n+2)} \cdot \frac{4n}{n+1}.$$

   The product of the last two terms is at most one because it simplifies to $2n/(2n+1)$, completing the inductive step.

   (d) The shape $C_n$ is an equilateral triangle of area $4^n$ is divided into congruent pieces of area 1 with one of the corners removed. $C_n$ can be tiled using the trapezoidal tile $T$ below. *(Source: Daniel J. Velleman. How to Prove it: A structured approach (3rd edition). Exercise 6.2.13)*

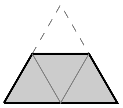


Shape $C_2$



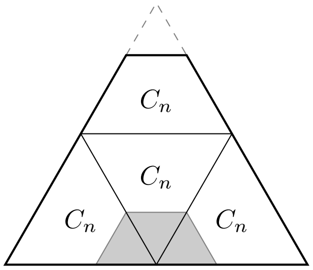
Tile $T$

**Solution: Base case:** $C_1$ has the shape of a single tile $T$ and can be tiled by it.



tiling of $C_1$

**Inductive step:** Assume $C_n$ can be tiled by copies of $T$. We show that so can $C_{n+1}$. Partition the triangle of area $4^{n+1}$ into four similar triangles of area $4^n$ as in the sketch below. Shape $C_{n+1}$ is derived by removing the top corner of the large triangle. It also covers the top corner of one of the four smaller triangles. Cover a corner of the other three using tile $T$. The uncovered parts comprise four copies of $C_n$. They can be tiled by the inductive hypothesis. The combined tiling covers $C_{n+1}$.



shape $C_{n+1}$

2. Use strong induction to prove the following for all positive integers $n \geq 1$.

   (a) If $n \geq 14$ then $n$ can be written as $4a + 5b$ for some integers $a, b \geq 0$.

   (b) $(3/2)^{n-2} \leq F_n \leq (7/4)^{n-2}$ for every $n \geq 4$, where $F_n$ is the $n$-th Fibonacci number ($F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$.)

   (c) $G_n = n!$, where $G_n$ is given by $G_1 = 1$, $G_{n+1} = 1 + G_1 + 2G_2 + \cdots + nG_n$. (**Hint:** $n \cdot n! = (n+1)! - n!$).

   **Solution:**

   (a) We first check that the predicate is true for $n = 14$, 15, 16 and 17:

$$\begin{aligned}14 &= 4 \cdot 1 + 5 \cdot 2\\ 15 &= 4 \cdot 0 + 5 \cdot 3\\ 16 &= 4 \cdot 4 + 5 \cdot 0\\ 17 &= 4 \cdot 3 + 5 \cdot 1.\end{aligned}$$

   Now we prove that it is true for all $n \geq 14$ by strong induction. We assume that the predicate is true for all values from 14 up to $n$. If $n+1$ is 15, 16, or 17 it was already checked. If $n + 1 > 17$, by our hypothesis we know that $(n + 1) - 4 = 4a + 5b$ for some $a, b \geq 0$, so $n + 1 = 4(a + 1) + 5b$.

   (b) **Base case** $n = 4$**:** $(3/2)^{4-2} = 9/4 \leq F_4 = 3 \leq (7/4)^{4-2} = 49/16$.
   **Inductive step:** Now assume that $(3/2)^{k-2} \leq F_k \leq (7/4)^{k-1}$ for $k$ ranging from 1 up to $n$. It is easier to manage the two inequalities separately. For the upper bound,

$$F_{n+1} = F_n + F_{n-1} \leq (7/4)^{n-2} + (7/4)^{n-3} = (7/4)^{n-1}(4/7 + (4/7)^2) = (7/4)^{n-1} \cdot (44/49) \leq (7/4)^{n-1}.$$

The first inequality is our use of the inductive hypothesis. For the lower bound,

$$F_{n+1} = F_n + F_{n-1} \geq (3/2)^{n-2} + (3/2)^{n-3} = (3/2)^{n-1}(2/3 + (2/3)^2) = (3/2)^{n-1} \cdot (10/9) \geq (3/2)^{n-1}.$$

The first inequality is again our use of the inductive hypothesis.

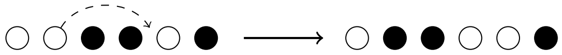
(c) **Base case:** $n = 1$ $G_1 = 1 = 1!$. So the claim holds for $n = 1$.

**Inductive step:** Now, assume $G_n = n!$ for $k = 1, 2$ up to $n$. Let $k = n + 1$. Then

$$\begin{aligned} G_{n+1} &= 1 + G_1 + 2G_2 + \cdots + nG_n \\ &= 1 + 1! + 2(2!) + \cdots + n(n!) \\ &= 1 + (2! - 1!) + (3! - 2!) + \cdots + ((n+1)! - n!) \\ &= 1 - 1! + 2! - 2! + 3! + \cdots - n! + (n+1)! \end{aligned}$$

which equals $(n + 1)!$ as desired because all other terms cancel out.

3. $n$ white pegs and $n$ black pegs are arranged in a line. In each step you are allowed to move any peg past *two* consecutive pegs of the opposite color, left or right. Initially all white pegs are to the left of the black ones.



(a) Assume $n$ is odd. Say a pair of pegs is *inverted* if one is black, one is white, and the black one is to the left of the right one. Prove that "the number of inverted pairs is even" is an invariant.

**Solution:** This is initially true as the number of inverted pairs is zero. Now assume it is true after $t$ steps. In step $t + 1$, the number of inverted pairs goes up by two if a white peg jumps to the right or a black peg jumps to the left, or down by two if a white peg jumps to the left or a black peg jumps to the right. In all cases, the number of inverted pairs stays even.

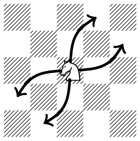
(b) If $n$ is odd, can the colors be reversed so that all black pegs are to the left of all white ones?

**Solution:** No. In the final configuration, every one of the $n^2$ black-white pairs is inverted. Since $n$ is odd, $n^2$ is also odd so there is an odd number of inverted pairs. Owing to the invariant from part (a) the final configuration can never be reached

(c) If $n$ is even, can the colors be reversed?

**Solution:** Yes. More generally we show by induction on $k$ that this is true for any number $k$ of white pegs and $n$ black pegs (as long as $n$ is even). When $k = 0$ there are no white pegs so there is nothing to reverse. Now we assume $k$ white pegs and $n$ black pegs can be reversed. Given $k + 1$ white pegs and $n$ black pegs, move the rightmost white peg to the right end by jumping two black pegs at a time and leave it there. By inductive hypothesis the remaining $k + n$ pegs can be reversed, so the whole configuration can be reversed.

4. A knight jumps around an infinite chessboard. Owing to injury it can only make these four moves:



(a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

**Solution:** The states are integer pairs $(x, y)$. The start state is $(0, 0)$. The transitions out of $(x, y)$ are

$$(x, y) \to (x - 2, y - 1) \quad \text{or} \quad (x - 1, y - 2) \quad \text{or} \quad (x + 2, y + 1) \quad \text{or} \quad (x + 1, y + 2).$$

(b) Prove that the knight can never reach any one of the four squares adjacent to the initial one by formulating a suitable invariant.

**Solution:** "3 divides $x+y$" is an invariant. It holds initially, and after every transition $(x, y) \to (x', y')$ we have $x'+y' = x+y-3$ in the first two cases and $x'+y' = x+y+3$ in the other two. Assuming the invariant holds before the transition (i.e., 3 divides $x + y$) it also holds after the transition (3 divides $x' + y'$).

The invariant does not hold for any of the four squares next to the initial one. $x + y$ takes value $-1$ or $+1$ on those squares. 3 does not divide $-1$ or $+1$. Therefore none of these four squares are reachable.

(c) Can the knight ever end up six squares to the right of the origin?

**Solution:** Yes. The fact that the invariant *does* hold for this square is not sufficient to prove this. The following sequence of transitions does.

$$(0, 0) \to (2, 1) \to (4, 2) \to (3, 0) \to (5, 1) \to (7, 2) \to (6, 0).$$