

1. Find exact closed-form solutions to the following recurrences in two ways (by unwinding and by homogenization), and verify the result by induction.

(a) $f(n) = 4f(n-1) + 9, f(0) = 1.$

Solution: *By unwinding:*

$$\begin{aligned} f(n) &= 4f(n-1) + 9 \\ &= 4(4f(n-2) + 9) + 9 = 4^2 \cdot f(n-2) + 4 \cdot 9 + 9 \\ &= 4^2(4f(n-3) + 9) + 4 \cdot 9 + 9 = 4^3 \cdot f(n-3) + 4^2 \cdot 9 + 4 \cdot 9 + 9 \end{aligned}$$

Continuing in this manner we get

$$f(n) = 4^n \cdot f(0) + 4^{n-1} \cdot 9 + \dots + 4 \cdot 9 + 9.$$

Plugging in $f(0) = 1$, we get

$$\begin{aligned} f(n) &= 4^n + 4^{n-1} \cdot 9 + \dots + 4 \cdot 9 + 9 \\ &= 4^n + 9 \cdot \frac{4^n - 1}{4 - 1} \\ &= 4^n + 3(4^n - 1) = 4^{n+1} - 3. \end{aligned}$$

By homogenization: We look for a constant c so that $f(n) + c = 4(f(n-1) + c)$. This c must satisfy $c + 9 = 4c$, so $c = 3$. The function $g(n) = f(n) + 3$ then satisfies the recurrence $g(n) = 4g(n-1)$ with initial condition $g(0) = f(0) + 3 = 4$. This gives $g(n) = 4^{n+1}$. Then $f(n) = g(n) - 3 = 4^{n+1} - 3$.

Verification by induction: We prove our guess is correct by induction on n . When $n = 0$, both $f(0)$ and $4^{n+1} - 3$ are 1. Now assume $f(n) = 4^{n+1} - 3$ for some n . Then

$$f(n+1) = 4f(n) + 9 = 4(4^{n+1} - 3) + 9 = 4^{n+2} - 12 + 9 = 4^{n+2} - 3$$

as it should be.

(b) $f(n) = \frac{3}{5}f(n-1) + \frac{4}{5}, f(0) = 0.$

Solution: *By unwinding:*

$$\begin{aligned} f(n) &= \frac{3}{5}f(n-1) + \frac{4}{5} \\ &= \frac{3}{5} \left(\frac{3}{5}f(n-2) + \frac{4}{5} \right) + \frac{4}{5} = \left(\frac{3}{5} \right)^2 f(n-2) + \frac{3}{5} \cdot \frac{4}{5} + \frac{4}{5} \\ &= \left(\frac{3}{5} \right)^2 \left(\frac{3}{5}f(n-3) + \frac{4}{5} \right) + \frac{3}{5} \cdot \frac{4}{5} + \frac{4}{5} = \left(\frac{3}{5} \right)^3 f(n-3) + \left(\frac{3}{5} \right)^2 \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{4}{5} + \frac{4}{5}. \end{aligned}$$

Continuing in this manner we get

$$f(n) = \left(\frac{3}{5} \right)^n \cdot f(0) + \left(\frac{3}{5} \right)^{n-1} \cdot \frac{4}{5} + \dots + \left(\frac{3}{5} \right)^2 \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{4}{5} + \frac{4}{5}.$$

Since $f(0) = 0$,

$$\begin{aligned} f(n) &= \left(\frac{3}{5} \right)^{n-1} \cdot \frac{4}{5} + \dots + \left(\frac{3}{5} \right)^2 \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{4}{5} + \frac{4}{5} \\ &= \frac{\frac{4}{5} \left(\left(\frac{3}{5} \right)^n - 1 \right)}{\frac{3}{5} - 1} \\ &= 2 - 2 \left(\frac{3}{5} \right)^n. \end{aligned}$$

By homogenization: We look for a constant c so that $f(n) + c = \frac{3}{5}(f(n-1) + c)$ for all n . This c must satisfy $c + \frac{4}{5} = \frac{3}{5}c$, so it equals -2 . We obtain the homogeneous recurrence $g(n) = \frac{3}{5}g(n-1)$ for $g(n) = f(n) - 2$. The initial condition is $g(0) = f(0) - 2 = -2$. This gives $g(n) = -2\left(\frac{3}{5}\right)^n$. Then $f(n) = g(n) + 2 = 2 - 2\left(\frac{3}{5}\right)^n$.

Verification by induction: We prove our guess is correct by induction on n . When $n = 0$, both $f(0)$ and $2 - 2\left(\frac{3}{5}\right)^0$ are 0. Now assume $f(n) = 2 - 2\left(\frac{3}{5}\right)^n$ for some n . Then

$$f(n+1) = \frac{4}{5} + \frac{3}{5}f(n) = \frac{4}{5} + \frac{3}{5}\left(2 - 2\left(\frac{3}{5}\right)^n\right) = \frac{4}{5} + \frac{6}{5} - \frac{6}{5}\left(\frac{3}{5}\right)^n = 2 - 2\left(\frac{3}{5}\right)^{n+1}$$

as it should be.

- (c) $f(n) = 3f(n/2) + n$, $f(1) = 1$, where n is a power of 2.

Solution:

By unwinding: We try to guess a solution by unwinding the formula for $f(n)$:

$$\begin{aligned} f(n) &= 3f\left(\frac{n}{2}\right) + n \\ &= 3\left(3f\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n = 3^2f\left(\frac{n}{2^2}\right) + 3 \cdot \frac{n}{2} + n \\ &= 3^2\left(3f\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + 3 \cdot \frac{n}{2} + n = 3^3f\left(\frac{n}{2^3}\right) + 3^2 \cdot \frac{n}{2^2} + 3 \cdot \frac{n}{2} + n \end{aligned}$$

Continuing in this manner for $\log n$ steps suggests the guess

$$f(n) = 3^{\log n} f(1) + 3^{\log n-1} \cdot \frac{n}{2^{\log n-1}} + 3^{\log n-2} \cdot \frac{n}{2^{\log n-2}} + \cdots + n.$$

By $f(1) = 1$ and geometric sum formula, we can write

$$f(n) = 3^{\log n} + \frac{(3/2)^{\log n} - 1}{(3/2) - 1} \cdot n = 3 \cdot 3^{\log n} - 2n = 3 \cdot n^{\log 3} - 2n.$$

By homogenization: We try the change of function $g(n) = f(n) + an$. For g to satisfy the homogenized recurrence $g(n) = 3g(n/2)$ we need $f(n) + an$ to equal $3f(n/2) + (3a/2)n$ for all n , so $3f(n/2) + n + an$ must equal $3f(n/2) + (3a/2)n$. Thus $a + 1$ must equal $3a/2$, from where a equals 2. The recurrence in g solves to $g(n) = n^{\log 3}g(1) = n^{\log 3}(f(1) + 2) = 3n^{\log 3}$. Therefore $f(n) = g(n) - 2n = 3 \cdot n^{\log 3} - 2n$.

- (d) *Verification by induction:* If we write $m = \log n$, then $f(2^m) = 3^{m+1} - 2^{m+1}$. We prove this is correct by induction on m . When $m = 0$, both $f(2^m)$ and $3^{m+1} - 2^{m+1}$ are one. Now assume $f(2^m) = 3^{m+1} - 2^{m+1}$ for some m . Then

$$f(2^{m+1}) = 3f(2^m) + 2^{m+1} = 3 \cdot (3^{m+1} - 2^{m+1}) + 2^{m+1} = 3^{m+2} - 2^{m+2}$$

as it should be.

Verification by induction: Asymptotic answer using the Master theorem: This is a divide-and-conquer recurrence that splits a size- n “problem” into 3 size- $n/2$ ones. Since $n = n^1$ has a smaller power than $n^{\log 3}$, the solution $f(n)$ is $\Theta(n^{\log 3})$.

2. Find exact closed-form solutions to the following recurrences.

- (a) $f(n) = 8f(n-1) - 15f(n-2)$, $f(0) = 0$, $f(1) = 1$

Solution: This is a homogeneous linear recurrence, so we guess a solution of the form $f(n) = x^n$ for some nonzero x . If our guess is correct, x^n must equal $8x^{n-1} - 15x^{n-2}$ for all n , from where $x^2 = 8x - 15$. This quadratic equation has the two solutions $x_1 = 3$ and $x_2 = 5$. Any linear combination of x_1^n and

x_2^n also satisfies the recurrence. We look for a linear combination $f(n) = sx_1^n + tx_2^n$ that satisfies the additional requirements $f(0) = 0$ and $f(1) = 1$:

$$\begin{aligned} 0 &= f(0) = s + t \\ 1 &= f(1) = sx_1 + tx_2 = 3s + 5t. \end{aligned}$$

The unique solution to this system is $s = -1/2$ and $t = 1/2$. Therefore

$$f(n) = -\frac{1}{2} \cdot 3^n + \frac{1}{2} \cdot 5^n = \frac{5^n - 3^n}{2}$$

is the solution to our recurrence.

- (b) $f(n) = f(n-1) + f(n-2) + 1$, $f(0) = 0$, $f(1) = 1$
(Hint: Try homogenizing with $f(n) = g(n) + c$ for some constant c .)

Solution: We first homogenize the recurrence. By adding one to both sides we can write

$$f(n) + 1 = (f(n-1) + 1) + (f(n-2) + 1).$$

The function $g(n) = f(n) + 1$ then satisfies the recurrence $g(n) = g(n-1) + g(n-2)$ with initial conditions $g(0) = 1$ and $g(1) = 2$. This is the same as the recurrence from Section 4 of Lecture 7, but with different initial conditions. We can solve it using the same method. Alternatively we can reason like this. If we define $g(-1) = 1$ then the recurrence is still satisfied for all $n \geq 1$. Then the function $h(n) = g(n-1)$ satisfies the same recurrence but with initial conditions $h(0) = g(-1) = 1$ and $h(1) = g(0) = 1$. This is exactly the same as the recurrence in Lecture 7, so

$$h(n) = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n+1},$$

from where

$$g(n) = h(n-1) = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n+2},$$

and finally

$$f(n) = g(n) - 1 = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n+2} - 1.$$

3. Recall that a *saddle* in a table of numbers is an entry that is largest in its column and smallest in its row. In Lecture 2 we showed that every table can have at most one saddle. Here is an algorithm for finding it (if it exists):

Input: A $n \times n$ table T . Assume n is a power of two and all entries of T are distinct.

Algorithm Saddle(T):

If $n = 1$, output the (unique) entry in T .

Otherwise,

 Recursively run **Saddle**(T_i) on each of the four quadrants T_1, T_2, T_3, T_4 of T .

 Let s_i be the output of **Saddle**(T_i).

 Test if s_i is a saddle of T by comparing it to

 all numbers in its row and column *except* those in T_i .

 If one of s_1, s_2, s_3 , or s_4 passes the test, output it.

- (a) Show a sample run of **Saddle** on the following input T :

12	2	5	10
16	7	13	4
15	8	14	9
6	1	11	3

Solution: The algorithm recursively splits the instance into four quadrants

$$T_1 = \begin{matrix} 12 & 2 \\ 16 & 7 \end{matrix} \quad T_2 = \begin{matrix} 5 & 10 \\ 13 & 4 \end{matrix} \quad T_3 = \begin{matrix} 15 & 8 \\ 6 & 1 \end{matrix} \quad T_4 = \begin{matrix} 14 & 9 \\ 11 & 3 \end{matrix}.$$

In the next level of recursion each of T_1, T_2, T_3, T_4 is split into its four entries and each of them is output as a candidate saddle. For example, 12, 2, 16, 7 are all considered as candidate saddles in T_1 and compared to the other entry in their row and column. Only 7 survives and is output as the saddle of T_1 . Similarly, 8 and 9 are output as the saddles of T_3 and T_4 , respectively. T_2 does not have a saddle so the recursive call **Saddle**(T_2) does not produce one. At this stage the candidate saddles in the four quadrants are

$$\begin{array}{cc|cc} 12 & 2 & 5 & 10 \\ 16 & \underline{7} & 13 & 4 \\ \hline 15 & \underline{8} & 14 & \underline{9} \\ 6 & 1 & 11 & 3 \end{array}$$

Now each candidate saddle is compared to the other number in its row and column except the ones in the table it is already in. Thus 7 is compared to 8, 1, 13, and 4, and so on. 7 does not survive because it is larger than 4 (and smaller than 8). Similarly 9 does not survive because it is smaller than 10. Only 8 survives and is the output produced by **Saddle**(T).

- (b) Let $C(n)$ be the worst-case number of comparisons **Saddle** performs on an $n \times n$ input. Explain why

$$C(n) \leq 4C(n/2) + 4n. \tag{1}$$

Solution: The number of comparisons $C(n)$ on a $n \times n$ table equals is the number of comparisons made by the four recursive calls, each of which is at most $C(n/2)$, plus the extra comparisons done in the test step. In general there can be as many as four candidate saddles coming in from the recursion. Each of them is compared to at most $n/2$ numbers in its row and $n/2$ numbers in its column. (Not all these comparisons are necessarily distinct; for example in part (a) 7 is compared to 8 twice, separately when each is considered as a possible candidate saddle.) So the number of extra comparisons is at most $4 \cdot (n/2 + n/2) = 4n$.

- (c) Apply Theorem 6 from Lecture 7 to calculate the big-Oh asymptotic growth of $C(n)$.

Solution: We instantiate the Master Theorem with $a = 4$, $b = 2$, and $g(n) = 4n$. Then $c = \log_b a = 2$, and $g(n) = O(n) = O(n^{c-\epsilon})$ with $\epsilon = 1$. So we have $C(n) = O(n^c) = O(n^2)$.

- (d) Obtain an exact formula for $C(n)$ assuming the inequality in (1) is an equality. Argue that your solution is an *upper bound* on the number of comparisons performed by **Saddle**.

Solution: We solve the recurrence $C(n) = 4C(n/2) + 4n$ with $C(1) = 0$. The solution to (1) can only be smaller by strong induction on n . We unwind the recurrence to obtain

$$\begin{aligned} C(n) &= 4C(n/2) + 4n \\ &= 4(4C(n/2^2) + 4(n/2)) + 4n = 4^2C(n/2^2) + 4^2n/2 + 4n \\ &= 4^2(4C(n/2^3) + 4(n/2^2)) + 4^2n/2 + 4n = 4^3C(n/2^3) + 4^3n/2^2 + 4^2n/2 + 4n \\ &= \dots \\ &= 4^{\log n}C(1) + 4 \cdot (n + 2n + 2^2n + \dots + 2^{\log n - 1}n) \\ &= 4(2^{\log n} - 1)n \\ &= 4n(n - 1). \end{aligned}$$

Assuming the equality that $C(n) = 4C(n/2) + 4n$ then $C(n) = \Theta(n^2)$ by the Master Theorem, Lecture 7. At the return of each recursive call to **Saddle**, for an input of size larger than $n = 1$, there is between 0 and 4 candidate saddles to consider. Hence the expression for $C(n)$ is a worst case scenario when at the return of each recursive call, the relevant comparisons for 4 saddles are made, and so the expression for $C(n)$ is an upper bound.

4. DNA (Deoxyribonucleic acid) is a molecule that carries the genetic instructions for all known organisms and many viruses. It consists of a chain of bases. In DNA chain, there are four types of bases: **A**, **C**, **G**, **T**. For example, a DNA chain of length 10 can be **ACGTACGTAT**.

(a) Let $g(n)$ be the number of configurations of a DNA chain of length n in which the pairs **TT** and **TG** never appear. Write a recurrence for $g(n)$. (**Hint:** Is the first base a **T**?)

Solution: A sequence of length n will either start with a **T** or not. If it starts with a **T**, then the next base must be either **C** or **A**. Then, any valid sequence of length $n - 2$ can be appended. Thus there are $2 \cdot g(n - 2)$ valid sequences of length n that start with **T**. If we do not start with **T**, then we have 3 choices and can append any valid sequence of length $n - 1$. Thus there are $3 \cdot g(n - 1)$ choices that do not start with **T**. Together we have the recurrence: $g(n) = 3g(n - 1) + 2g(n - 2)$. As base cases, we have $g(1) = 4$ and $g(2) = 14$.

(b) Solve the recurrence from part (a).

Solution: This is a homogeneous linear recurrence, so we guess a solution of the form $g(n) = x^n$ for some nonzero x . If our guess is correct, x^n must equal $3x^{n-1} + 2x^{n-2}$ for all n , from where $x^2 = 3x + 2$. This quadratic equation has the two solutions $x_1 = \frac{3+\sqrt{17}}{2}$ and $x_2 = \frac{3-\sqrt{17}}{2}$. Any linear combination of x_1^n and x_2^n also satisfies the recurrence. We look for a linear combination $g(n) = sx_1^n + tx_2^n$ that satisfies the additional requirements $g(0) = 1$ and $g(1) = 4$:

$$1 = g(0) = s + t$$

$$4 = g(1) = sx_1 + tx_2 = s \cdot \frac{3 + \sqrt{17}}{2} + t \cdot \frac{3 - \sqrt{17}}{2}.$$

The unique solution to this system is $s = \frac{17+5\sqrt{17}}{34}$ and $t = \frac{17-5\sqrt{17}}{34}$. Therefore

$$g(n) = \frac{17 + 5\sqrt{17}}{34} \left(\frac{3 + \sqrt{17}}{2} \right)^n + \frac{17 - 5\sqrt{17}}{34} \left(\frac{3 - \sqrt{17}}{2} \right)^n$$

is the solution to our recurrence.

(c) Which one of the alternatives $g(n) = o(3^n)$, $g(n) = \Theta(3^n)$, or $3^n = o(g(n))$ is correct?

Solution: The last one. $(3 - \sqrt{17})/2$ is about -0.562 so $\left(\frac{3-\sqrt{17}}{2}\right)^n$ tends to 0 as n gets large. Therefore $g(n)$ is $\Theta\left(\left(\frac{3+\sqrt{17}}{2}\right)^n\right)$. As $(3 + \sqrt{17})/2 > (3 + \sqrt{9})/2 = 3$, 3^n is $o\left(\left(\frac{3+\sqrt{17}}{2}\right)^n\right)$ so it must be $o(g(n))$ as well.