

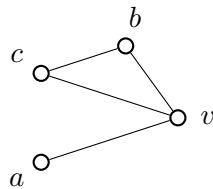
1. Are the following propositions about graphs true or false? Justify your answer. Specify your proof method.

- (a) Assume  $G$  is connected. Let  $G'$  be the graph obtained by removing an edge  $e$  from  $G$ .  $G'$  is connected if and only if  $e$  belongs to a cycle in  $G$ .

**Solution:** True. If edge  $e = \{u, v\}$  lies on a cycle in  $G$  then there is a path  $p$  from  $u$  to  $v$  that does not contain  $e$ . Since  $G$  is connected, there is a path between every two vertices. If  $e$  occurs on this path, it can be replaced by  $p$  to obtain a path in  $G'$ , so the graph remains connected. Conversely, if  $e$  does not lie on a cycle then  $G'$  cannot be connected because if  $G'$  has a path  $p$  from  $u$  to  $v$  then  $p$  together with  $e$  would form a cycle in  $G$ , a contradiction.

- (b) Assume  $G$  is connected. Let  $G'$  be the graph obtained by removing a vertex  $v$  and its incident edges from  $G$ .  $G'$  is connected if and only if  $v$  belongs to a cycle in  $G$ .

**Solution:** False. This graph below becomes disconnected after removing  $v$ :



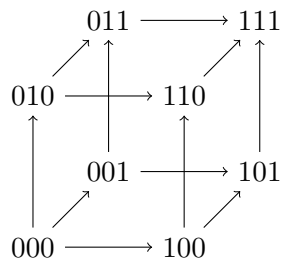
- (c) If every vertex in  $G$  belongs to a closed walk of odd length then there are at least as many edges as there are vertices in  $G$ .

**Solution:** True. We first prove this under the assumption that  $G$  is connected. If  $G$  has an odd-length closed walk then it must have an odd-length cycle. If  $G$  has a cycle then it is not a tree. Since  $G$  is connected it has more edges than any spanning tree of it. As a tree has one edge less than it has vertices,  $G$  must have at least as many edges as it has vertices. In general, applying this reasoning to every connected component of  $G$  we conclude that the claim must be true for  $G$ .

2. The *directed hypercube*  $D_n$  is the graph whose  $2^n$  vertices are the  $\{0, 1\}$  strings of length  $n$ . There is a directed edge from  $u$  to  $v$  if in some position,  $u$  is zero and  $v$  is one and they are identical in the other positions. For example,  $001 \rightarrow 011$  is an edge in  $D_2$  but there is no edge from  $011$  to  $001$ .

- (a) Draw the graph  $D_3$ .

**Solution:**



- (b) Find a topological sort of  $D_3$ .

**Solution:** 000, 001, 010, 100, 110, 101, 011, 111 is a topological sort.

- (c) Find a parallel schedule of  $D_3$  of minimum duration. How long is the longest path in  $D_3$ ?

**Solution:**  $V_0 = \{000\}$ ,  $V_1 = \{001, 010, 100\}$ ,  $V_2 = \{011, 101, 110\}$ ,  $V_3 = \{111\}$  is a parallel schedule of duration 4. The longest paths have length 3, e.g., 000, 001, 011, 111.

(d) Repeat part (c) for  $D_n$ .

**Solution:** If  $V_k$  are the  $n$ -bit strings with exactly  $k$  ones, then  $V_0, V_1, \dots, V_n$  is a parallel schedule of duration  $n + 1$ . There is a path of length  $n$  obtained by starting at  $00 \cdots 0$  and flipping the bits to 1 one at a time, say from left to right. Therefore no parallel schedule of duration less than  $n + 1$  is possible.

3. Let  $G$  be the digraph whose vertices are the 125 3-digit numbers with digits 1, 2, 3, 4, 5, and  $(u, v)$  is an edge if  $v - u$  equals 1, 10, or 100.

(a) Show that  $G$  is acyclic.

**Solution:** Order the numbers from smallest to largest. This ordering is a topological sort: There can be no back edges because the differences in the backward direction are all negative.

(b) What is the length of the longest path in  $G$ ? Justify your answer.

**Solution:**  $G$  has a path of length 12:

$$(111, 112, 113, 114, 115, 125, 135, 145, 155, 255, 355, 455, 555).$$

On the other hand,  $G$  cannot have a path of length 13 because along every edge at least one of the digits must increase, and this can happen at most four times for each digit.

(c) Use part (b) to show that  $G$  must have an antichain of size 10.

**Solution:** As  $G$  does not have a path of length 13, by Corollary 11 from Lecture 9 it must have an antichain of length at least  $125/13 > 9$ .

(d) **(Optional)** Show that  $G$  has an antichain of size 19.

**Solution:** If there is a path from  $v$  to  $w$  then the sum of digits of  $w$  must be greater than the sum of digits of  $v$ . Therefore the set of vertices whose digits sum up to 9 is an antichain. There are 19 such vertices.

(e) **(Optional)** Show that the vertices of  $G$  can be partitioned into 19 (vertex-disjoint) paths. Conclude that  $G$  cannot have an antichain of size 20.

**Solution:** The following 19 paths are vertex-disjoint and partition all 125 vertices:

$$\begin{aligned} p_a &= (a11, a12, a13, a14, a15, a25, a35, a45, a55), & \text{where } a \in \{1, 2, 3, 4, 5\} \\ q_a &= (a21, a22, a23, a24, a34, a44, a54), & \text{where } a \in \{1, 2, 3, 4, 5\} \\ r_{bc} &= (1bc, 2bc, 3bc, 4bc, 5bc), & \text{where } b \in \{3, 4, 5\} \text{ and } c \in \{1, 2, 3\} \end{aligned}$$

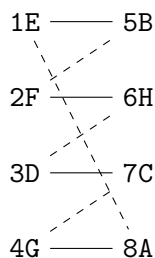
If  $G$  had an antichain of size 20, then two of its vertices would have to belong to one of these 19 paths. However, no two vertices in an antichain can belong to the same path.

4. In this question you will work out vertex-disjoint paths for the following source-sink pairs in the Beneš network  $B_3$ . The sources are labeled 1 to 8 and the sinks are labeled A to H from top to bottom.

1E 2F 3D 4G 5B 6H 7C 8A

(a) For each source-sink pair above, determine whether the path should be routed through the top or through the bottom.

**Solution:** The “conflict graph” looks like this, with solid and dashed edges showing conflicts between sources and sinks, respectively:



This graph is bipartite with respect to the partition  $T = \{1E, 2F, 3D, 4G\}$  and  $B = \{5B, 6H, 7C, 8A\}$ .

- (b) Route the top and bottom paths from part (a) recursively. Draw a diagram of the resulting eight vertex-disjoint paths.

**Solution:** The “constraint graphs” for the top and bottom copies of  $B_2$  are



We can route 1E and 2F through the top part of the top copy of  $B_2$ , 3D and 4G through the bottom part, 5B and 8A through the top part of the bottom copy of  $B_2$ , and 6H and 7C through the bottom part of the bottom copy. This yields the following vertex-disjoint paths:

