1. Find and explain the mistakes in the following "proofs".

   (a) **Theorem:** In every group of 5 people there is a person who is friends with at least 3 of them.

   *Proof.* Let Alice and Bob be two of the five people. The proof is by case analysis. We consider two cases:

   - **Case 1:** Alice is friends with at least 3 other people in the group.
   - **Case 2:** Bob is friends with at least 3 other people in the group.

   It follows that at least one of Alice and Bob is friends with at least 3 other people, so a person that is friends with at least 3 others always exists. $\square$

   (b) **Theorem:** Every group of 8 people includes a group of 4 friends or a group of 4 strangers.

   *Proof.* Let Alice be one of the eight people. The proof is by case analysis. We consider two cases:

   - **Case 1:** Alice is friends with at least 4 other people in the group.
   - **Case 2:** Alice is a stranger to at least 4 other people in the group.

   One of these two cases must hold. Let's discuss Case 1. If all the people who are friends with Alice are strangers among themselves, this is a group of 4 strangers. Otherwise, at least 3 of them are mutual friends, and together with Alice they form a group of 4 friends.

   Now let's do Case 2. If all the people who are strangers to Alice are friends among themselves, this is a group of 4 friends. Otherwise, at least 3 of them are mutual strangers, and together with Alice they form a group of 4 strangers. $\square$

   (c) **Theorem:** In every 3 by 3 table containing the digits `1` to `9` each once, some two consecutive digits must appear in the same row or in the same column.

   *Proof.* We prove this by contradiction. Once we fix the placement of `1` there are four positions from which `2` is blocked, namely the two in the same column and the two in the same row. When we position `2` in one of the remaining ones, there are now four more positions from which `3` is blocked. We repeat the argument one more time. There are now a total of $3 \times 4 = 12$ blocked positions so `4` cannot be placed anywhere in the table. This is a contradiction so a table with the desired properties cannot exist. $\square$

2. Prove the following theorems using the suggested strategy.

   (a) For every integer $n$ which is not a multiple of 3, $n^4 + n^2 + 1$ is a multiple of 3. *Proof by cases*

   (b) For every real number $x$ such that $-1 \leq x \leq 1$, $x^3 - x^2 + x < 2$. *Forward proof of implication*

   (c) For all positive real numbers $x, y$, if $x$ is irrational, at least one of the numbers $x^2 - y^2$ and $x + y$ is irrational. *Proof by contrapositive*

3. Prove the following theorems. (If you claim a number is irrational, provide a proof or give a reference, for example "By Theorem 9 in Lecture 2, $\sqrt{2}$ is irrational.")

   (a) $\sqrt[3]{2}$ is an irrational number.

   (b) $\sqrt{2} + \sqrt{5}$ is an irrational number.

   (c) **(Optional)** $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is an irrational number.

4. *[Adapted from Rosen Exercise 1.8.1]* Four `0`s and Five `1`s are arranged around a circle in some order. You replace each consecutive pair of bits with a `0` if they are equal and with a `1` if they are different, obtaining another configuration of nine bits. After some number of repetitions can you end up with

   (a) One `1` and eight `0`s?

   (b) Eight `1`s and one `0`?

   (c) **(Mini-project, optional)** $n$ `1`s and $(9 - n)$ `0`s for other values of $n$?

   Prove that your answer is correct.