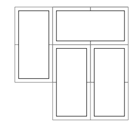
CSI 2101 B/C: Discrete Structures — Homework 3
University of Ottawa, Winter 2023

1. Prove the following using induction. n is a positive integer.
   (a) For every n, the sum of the first n odd integers is n^2.
   (b) For every n, $1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.
   (c) For every odd n, an $n \times n$ grid with a corner square removed can be tiled using 2×1 pieces. (See example for $n = 3$. Your proof may describe a different tiling.)



2. Use strong induction to prove the following for all positive integers $n \geq 1$.
   (a) $(3/2)^{n-2} \leq F_n \leq (7/4)^{n-2}$ for every $n \geq 4$, where F_n is the n-th Fibonacci number ($F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$.)
   (b) $G_n = n!$, where G_n is given by $G_1 = 1$, $G_{n+1} = 1 + G_1 + 2G_2 + \cdots + nG_n$. (**Hint:** $n \cdot n! = (n+1)! - n!$)
   (c) [Rosen, Exercise 5.12] n is a sum of distinct powers of 2, e.g., $5 = 2^2 + 2^0$.

3. Let $P(n)$ be the predicate "In every group of $n(n+1)/2$ people there are n mutual friends or 3 mutual strangers."
   (a) Prove $P(2)$.
   (b) Let $Q(n)$ be the predicate "there exist n people in the group who are all strangers to Alice." (Alice is some person in the group.) Assume $Q(n)$ is true. Prove that $P(n)$ must be true.
   (c) Now assume $P(n-1)$ and NOT $Q(n)$ are both true. Prove that $P(n)$ must be true.
   (d) Prove that $P(n)$ is true for all n.

4. You have a system of n switches, each of which can be in one of two states: OFF or ON. There are 2^n possible configurations of this system. For example, when $n = 2$ the four possible configurations for the pair of switches are (OFF, OFF), (OFF, ON), (ON, OFF), and (ON, ON).

   Initially, all switches are OFF. In each step, you are allowed to flip exactly one of the switches. Is there a sequence of flips that makes each possible configuration arise exactly once? For example when $n = 2$ this sequence of flips has the desired property (the number on the arrow indicates the switch that is flipped):

$$(\text{OFF}, \text{OFF}) \xrightarrow{2} (\text{OFF}, \text{ON}) \xrightarrow{1} (\text{ON}, \text{ON}) \xrightarrow{2} (\text{ON}, \text{OFF})$$

   (a) Show a sequence of flips that works when $n = 3$.
   (b) Prove that for every $n \geq 1$, there exists a sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration. (**Hint:** Use induction. You may need to strengthen the proposition.)
   (c) Now suppose that you flip not one but two switches at a time. Prove that for every $n \geq 2$, there is no sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration. (**Hint:** Use an invariant.)