1. Find exact closed-form solutions to the following recurrences.
   (a) $T(n) = 3T(n/2) + n$, $T(1) = 1$, where n is a power of 2.
   (b) $F(n) = \frac{1}{3}F(n-1) + 1$, $F(0) = 0$.
   (c) $f(n) = 8f(n-1) - 15f(n-2)$, $f(0) = 0$, $f(1) = 1$
   (d) $f(n) = f(n-1) + f(n-2) + 1$, $f(0) = 0$, $f(1) = 1$
   (**Hint:** Try homogenizing with $f(n) = g(n) + c$ for some constant c.)

2. Recall that a *saddle* in a table of numbers is an entry that is largest in its column and smallest in its row. In Lecture 2 we showed that every table can have at most one saddle. Here is an algorithm for finding it (if it exists):

   Input: A $n \times n$ table T. Assume n is a power of two and all entries of T are distinct.
   Algorithm Saddle(T)**:**
   If $n = 1$, output the (unique) entry in T.
   Otherwise,
   Recursively run **Saddle**(T_i) on each of the four quadrants T_1, T_2, T_3, T_4 of T.
   Let s_i be the output of **Saddle**(T_i).
   Test if s_i is a saddle of T by comparing it to
   all numbers in its row and column *except* those in T_i.
   If one of s_1, s_2, s_3, or s_4 passes the test, output it.

   (a) Show a sample run of **Saddle** on the following input T:

12	2	5	10
16	7	13	4
15	8	14	9
6	1	11	3

   (b) Let $C(n)$ be the worst-case number of comparisons **Saddle** performs on an $n \times n$ input. Explain why

$$C(n) \leq 4C(n/2) + 4n. \quad (1)$$

   (c) Apply Theorem 6 from Lecture 7 to calculate the big-Oh asymptotic growth of $C(n)$.
   (d) Obtain an exact formula for $C(n)$ assuming the inequality in (1) is an equality. Argue that your solution is an *upper bound* on the number of comparisons performed by **Saddle**.

3. DNA (Deoxyribonucleic acid) is a molecule that carries the genetic instructions for all known organisms and many viruses. It consists of a chain of bases. In DNA chain, there are four types of bases: **A**, **C**, **G**, **T**. For example, a DNA chain of length 10 can be **ACGTACGTAT**.
   (a) Let $g(n)$ be the number of configurations of a DNA chain of length n in which the pairs **TT** and **TG** never appear. Write a recurrence for $g(n)$. (**Hint:** Is the first base a **T**?)
   (b) Solve the recurrence from part (a).
   (c) Which one of the alternatives $g(n) = o(3^n)$, $g(n) = \Theta(3^n)$, or $3^n = o(g(n))$ is correct?

4. Bottle T has 4 litres of tea and bottle C has 4 litres of coffee. In each step you pour 1 litre of liquid from bottle T into bottle C, stir, pour back 1 litre of liquid from bottle C into bottle T. T, and stir again. (This is a recipe for yuanyang.) Let $f(n)$ be the amount of coffee in bottle T after n steps.

   (a) Calculate the values $f(0)$, $f(1)$, and $f(2)$.

   (b) Write a recurrence for $f(n)$.

   (c) Solve the recurrence from part (b).

   (d) What is limiting value of $f(n)$ as n approaches infinity? How many steps do you need to perform to approach this value within 0.01 litres?