Practice Final 1

1. In a group of 15 people, is it possible for each person to have exactly 3 friends? (If Alice is a friend of Bob we assume Bob is also a friend of Alice.)

2. Alice places two pebbles at the opposite corners of an 8 by 8 chessboard. At each step, she can

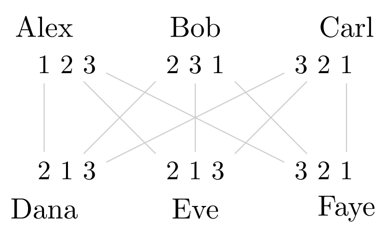
   - put a new pebble in an empty square, if *exactly one* of its neighbors contains a pebble, or
   - remove a pebble from a square, if *at least one* of its neighbors contains a pebble.

   Neighbors are squares that share a common side. Can the board ever have a single pebble on it?

3. Let a and b be integers. Show that if 3 is an integer combination of $2a$ and b and 5 is an integer combination of a and $2b$ then $\gcd(a, b) = 1$.

4. The vertices of graph G are the integers from 1 to 20. The edges of G are the pairs $\{x, y\}$ such that $\gcd(x, y) > 1$. How many connected components does G have?

5. Find a stable matching for these preferences and show that there is no other stable matching.



6. Pebbles ①···⑧ and ❶···❽ are placed on the sources and sinks of the Beneš network B_3, respectively, in arbitrary order. In each step, one pebble can be moved along an edge to an empty vertex, white ones forward and black ones backward. Can the positions of ⓧ and ● x be flipped for every x?

Practice Final 2

1. Let G be a graph with 10 vertices and 9 edges. Is it true that G must be a tree? Justify your answer.

2. A box contains 100 black balls and 99 white balls. In each step Alice takes out two balls of the same color and puts in one ball of the opposite color. Can Alice be left with exactly one ball of each colour in the box?

3. Show that for every two integers m and n, $(m + n)^3$ is even if and only if $m^3 + n^3$ is even.

4. What is the multiplicative inverse of 100 modulo 1009? Show your work.

5. Let G be the following graph. The vertices of G are all the integers between -10 and 10 except for 0 (20 vertices in total). The pair $\{x, y\}$ is an edge of G if (and only if) $-30 < xy < 0$.

   (a) Show that G is bipartite.

   (b) Show that G does not have a perfect matching.

6. A *cut-edge* in a connected graph is an edge e such that if e was removed, the graph would no longer be connected. Show that any connected graph in which all vertices have even degree does not have a cut-edge.

Practice Final 3

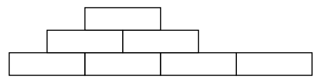
1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".

2. The sequence $f(n)$ is given by $f(n+1) = 2^{f(n)}$ for $n \geq 1$ with $f(0) = 2$.

   (a) Calculate $f(n)$ mod 5 for $n = 1$, $n = 2$, and $n = 3$.

   (b) Give a formula for $f(n)$ mod 5 for all $n \geq 4$. Justify your answer.

3. Blocks of height one are stacked in layers in some formation. Each layer has strictly fewer blocks than the one under it. For example the 7-block formation below has height 3. Show that the height of an n-block formation is $O(\sqrt{n})$.



4. Prove that every tree can have at most one perfect matching. Specify your proof method.

5. G is a directed graph whose vertices are the integers from -10 to 10 (inclusive) and whose edges (x, y) are those ordered pairs for which $|x| - |y| = 1$. For each of the following claims, say if it is true or false and provide a proof.

   (a) G has a path of length 10.

   (b) G has a parallel schedule of duration 11.

   (c) G has an antichain of size 6.

6. The vertices of graph H_n are the n integers from $-n$ to n except 0. The edges of H_n are the pairs $\{x, y\}$ such that $x = -y$ or $|y - x| = 1$.

   (a) Show that H_n is bipartite.

   (b) How many perfect matchings do H_1 and H_2 have?

   (c) How many perfect matchings does H_{10} have? (**Hint:** Write a recurrence.)