Practice Final 1

1. In a group of 15 people, is it possible for each person to have exactly 3 friends? (If Alice is a friend of Bob we assume Bob is also a friend of Alice.)

   Solution: No. Suppose for contradiction this was possible. Then the sum of degrees in the friendship graph would have been $15 \cdot 3 = 45$. But the sum of the degrees equals twice the number of edges, which is an even number, contradicting the fact that 45 is odd.

2. Alice places two pebbles at the opposite corners of an 8 by 8 chessboard. At each step, she can

   - put a new pebble in an empty square, if *exactly one* of its neighbors contains a pebble, or
   - remove a pebble from a square, if *at least one* of its neighbors contains a pebble.

   Neighbors are squares that share a common side. Can the board ever have a single pebble on it?

   Solution: No. Let G be the graph whose vertices are the pebbles and edges are pebbles that are neighbors. The predicate "G has at least two connected components" is an invariant of this state machine. Initially G has two connected components so the invariant holds. We show that the number of connected components cannot decrease after a transition. For the first type of move it remains the same because the new pebble extends an existing component without affecting the others. For the second type of move, the component that the removed pebble belongs to may break up up into one or more components, while the other components are unaffected, so the number of components cannot decrease. A board with a single pebble has one connected component so this state can never be reached.

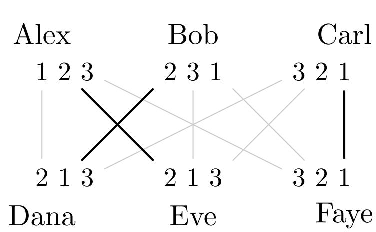
3. Let a and b be integers. Show that if 3 is an integer combination of $2a$ and b and 5 is an integer combination of a and $2b$ then $\gcd(a, b) = 1$.

   Solution: If 3 is an integer combination of $2a$ and b, then 3 is also an integer combination of a and b. Similarly, if 5 is an integer combination of a and $2b$, then 5 is also an integer combination of a and b. Integer combinations of integer combinations are also integer combinations, so $1 = 2 \cdot 3 - 5$ is also an integer combination of a and b. Since $\gcd(a, b)$ must divide all their integer combinations, $\gcd(a, b)$ divides 1, so it must be equal to 1.

4. The vertices of graph G are the integers from 1 to 20. The edges of G are the pairs $\{x, y\}$ such that $\gcd(x, y) > 1$. How many connected components does G have?

   Solution: Six. The vertex 1 is a connected component on its own. So is each of the vertices 11, 13, 17, and 19: These are all prime numbers that are greater than 10, so their gcd with any number between 1 and 20 equals one. We now argue that all the remaining vertices are in a single connected component. These vertices are the multiples of 2, 3, 5, and 7, so each remaining vertex is connected to 2, 3, 5, or 7. It remains to show that 2, 3, 5, and 7 are in the same connected component. This is true because G contains the paths $(2, 6, 3)$, $(2, 10, 5)$, and $(2, 14, 7)$.

5. Find a stable matching for these preferences and show that there is no other stable matching.



Solution: Consider the marked matching {Alex, Eve}, {Bob, Dana}, {Carl, Faye}. We show that no other matching is stable. As a stable matching always exists, this one must be stable.

In any stable matching, Carl must be matched to Faye because they are each other's first choice (so they would be a rogue couple if not matched). For the rest, the matching {Alex, Dana}, {Bob, Eve} can be ruled out because Bob and Dana would be a rogue couple. This leaves the above matching as the only stable possibility.

Alternative solution: If we run the Gale-Shapley algorithm, on day 1 Alex proposes to Dana and Bob and Carl propose to Faye. Faye picks Carl, so on day 2 both Alex and Bob propose to Dana. Dana picks Bob, so the final matching is {Alex, Eve}, {Bob, Dana}, {Carl, Faye}. We proved in Lecture 5 that this is stable.

Let us now run the Gale-Shapley algorithm again, but with the girls doing the proposing this time around. On day 1 Dana and Eve propose to Bob and Faye proposes to Carl. Carl picks Faye and Bob picks Dana over Eve. On day 2 Eve proposes to Alex resulting in the same final stable matching.

By Theorem 6 in Lecture 8, the first matching is the best possible for the boys (every boy gets his best possible choice among all stable matchings), while the second one is the worst possible for the boys (every boy gets his worst possible choice). Since they are the same there can be only one stable matching.

6. Pebbles ①⋯⑧ and ❶⋯❽ are placed on the sources and sinks of the Beneš network B_3, respectively, in arbitrary order. In each step, one pebble can be moved along an edge to an empty vertex, white ones forward and black ones backward. Can the positions of ⓧ and ● x be flipped for every x?

   Solution: Yes. It is sufficient to describe how to flip the positions of ⓧ and ● x while all the other pebbles occupy the remaining sources and sinks. Recall that the network B_3 consists of a top and bottom copy of B_2 plus 8 sources and sinks. Each source/sink has an outgoing/ incoming edge to/from both the top and bottom copies of B_2, respectively.

   Let v and v' be the initial positions of ⓧ and ● x, respectively. The network B_3 has edges (v, t), (v, b), (t', v'), and (b', v'), where t and t' belong to the top copy of B_2 and b and b' belong to the bottom copy of B_2, respectively. Since B_2 is a switching network, there must exist paths from t to t' and from b to b'. Then ⓧ and ● x can be flipped by the following sequence of moves: (1) Move ⓧ from v to t; (2) Move ● x from v' to b'; (3) Move ⓧ from t to t' along the path in the top copy of B_2 and on to v'; (4) Move ● x from b' to b along the path in the bottom copy of B_2 and on to v.

Practice Final 2

1. Let G be a graph with 10 vertices and 9 edges. Is it true that G must be a tree? Justify your answer.

   Solution: No. For example, the graph G that consists of one cycle on 9 vertices and one isolated vertex (two connected components) has 10 vertices and 9 edges but is not a tree.

2. A box contains 100 black balls and 99 white balls. In each step Alice takes out two balls of the same color and puts in one ball of the opposite color. Can Alice be left with exactly one ball of each colour in the box?

   Solution: No. We show that the predicate "3 divides $b - w - 1$" is an invariant of the underlying state machine, where b and w is the number of black and white balls respectively. The invariant holds initially. We now argue that it is preserved by the transitions, so assume 3 divides $b - w + 1$ before a given transition. There are two possibilities after the transition: Either the box contains $b - 2$ black and $w + 1$ white balls, in which case $(b - 2) - (w + 1) - 1 = (b - w - 1) - 3$ is a multiple of 3, or the box contains $b + 1$ black and $w - 2$ white balls, in which case $(b + 1) - (w - 2) - 1 = (b - w - 1) + 3$ is also a multiple of 3. Since 3 does not divide $1 - 1 - 1 = -1$, the state in which there is exactly one ball of each colour cannot be reached.

3. Show that for every two integers m and n, $(m + n)^3$ is even if and only if $m^3 + n^3$ is even.

   Solution: We prove this proposition by case analysis. The cases are summarized in the following table:

m	n	$m+n$	$(m+n)^3$	m^3	n^3	m^3+n^3
even	even	even	even	even	even	even
even	odd	odd	odd	even	odd	odd
odd	even	odd	odd	odd	even	odd
odd	odd	even	even	odd	odd	even

We used the fact that the cube of an integer has the same parity as the integer. We can see that $(m+n)^3$ and m^3+n^3 have the same parity in all cases.

Alternative solution: The two numbers have the same parity if and only if their difference $d = (m+n)^3 - (m^3+n^3)$ is even. The difference expands to $d = 3m^2n + 3mn^2 = 3mn(m+n)$. d contains m and n as factors so if either of them is even so is d. If both are odd then $m+n$ is even and so is d.

4. What is the multiplicative inverse of 100 modulo 1009? Show your work.

Solution: We look for numbers s and t such that $100s + 1009t = 1$. The extended Euclid's algorithm goes through the steps

$$1009 = 10 \cdot 100 + 9$$
$$100 = 11 \cdot 9 + 1,$$

from where

$$1 = 100 - 11 \cdot 9 = 100 - 11 \cdot (1009 - 10 \cdot 100) = 111 \cdot 100 - 11 \cdot 1009$$

so we can set $s = 111$ and $t = -11$. Therefore $100 \cdot 111 \equiv 1 \mod 1009$ and 111 is the desired multiplicative inverse.

5. Let G be the following graph. The vertices of G are all the integers between -10 and 10 except for 0 (20 vertices in total). The pair $\{x, y\}$ is an edge of G if (and only if) $-30 < xy < 0$.

(a) Show that G is bipartite.

Solution: G is bipartite with respect to the partition N, P with $N = \{-10, \ldots, -1\}$ and $P = \{1, \ldots, 10\}$: If x and y are both in P or both in N then $xy > 0$ so $\{x, y\}$ is not an edge of G.

(b) Show that G does not have a perfect matching.

Solution: To show that G has no perfect matching, we exhibit a subset S of P of size 5 whose neighbour set has size at most 4. By Hall's theorem, the vertices in S cannot all be matched. Take $S = \{6, 7, 8, 9, 10\}$. The vertices -5, -6, -7, -8, -9 and -10 are not neighbours of S since any product between one of this numbers and a number in S is at most $-5 \cdot 6 = -30$. Therefore S can have at most 4 neighbours.

6. A *cut-edge* in a connected graph is an edge e such that if e was removed, the graph would no longer be connected. Show that any connected graph in which all vertices have even degree does not have a cut-edge.

Solution: We prove this is impossible by contradiction. Suppose there exists a connected graph G with exactly one cut-edge $e = \{u, v\}$ in which all vertices have even degree. Let G' be the graph obtained by removing the edge e from G. Then u and v must belong to different connected components C and C' of G' (for otherwise removing e from G would not disconnect it.) All the vertices of C except for u have even degree, so C has exactly one vertex of odd degree. Therefore the sum of the degrees of the vertices in C is odd. This is impossible: In lecture 5 we showed that the sum of the degrees of all vertices in a graph (and therefore in all of its connected components) must be even.

Practice Final 3

1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".

   Solution: $\forall u, b_1, b_2 \colon IN(b_1, u)$ AND $IN(b_2, u) \longrightarrow b_1 = b_2$. Any two balls in any given urn must be the same ball.

2. The sequence $f(n)$ is given by $f(n+1) = 2^{f(n)}$ for $n \geq 1$ with $f(0) = 2$.

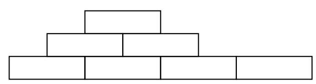
   (a) Calculate $f(n)$ mod 5 for $n = 1$, $n = 2$, and $n = 3$.

   Solution: $f(1) = 2^2 = 4$, $f(2) = 2^{f(1)} = 16$ so $f(2) \equiv 1 \pmod 5$, $f(3) = 2^{16}$. This is a pretty large number but we can calculate $2^{16} \pmod 5$ using Fermat's little Theorem: $2^{16} \equiv (2^4)^4 \equiv 1^4 \equiv 1 \pmod 5$.

   (b) Give a formula for $f(n)$ mod 5 for all $n \geq 4$. Justify your answer.

   Solution: $2^k \equiv 1 \pmod 5$ whenever k is a multiple of 4. As $f(n)$ is a multiple of 4 for every $n \geq 1$ (it is a product of many 2s) we get that we get that $f(n+1) = 2^{f(n)} \equiv 1 \pmod 5$ for all $n \geq 1$.

3. Blocks of height one are stacked in layers in some formation. Each layer has strictly fewer blocks than the one under it. For example the 7-block formation below has height 3. Show that the height of an n-block formation is $O(\sqrt{n})$.



   Solution: A formation of height k must have at least i blocks in its i-th level from the top because the number of blocks increases by one in each level. Therefore the number of blocks n must be at least $1 + 2 + \cdots + k = k(k+1)/2 > k^2/2$. Therefore $k < \sqrt{2n}$ which is $O(\sqrt{n})$.

4. Prove that every tree can have at most one perfect matching. Specify your proof method.

   Solution: The proof is by strong induction on the number of vertices. If a tree has one vertex then it has no perfect matching so the proposition holds. Now assume it is true for all trees with fewer than n vertices and consider any tree T with n vertices. T must have a vertex v of degree one. This vertex v can be matched in at most one way to its unique neighbor w. We now argue that there exists at most one matching that covers all remaining vertices. The graph G obtained by removing v and w from T with all their incident edges is a forest. By the inductive assumption, each connected component of G can have at most one perfect matching, so G itself, and therefore T also, can have at most one perfect matching.

   Alternative solution: We prove the contrapositive: A union of any two distinct perfect matchings Ξ_0 and Ξ_1 on the same set of vertices must contain a cycle, so Ξ_0 and Ξ_1 cannot both be perfect matchings of a tree. (Distinct does not mean *disjoint*: Ξ_1 and Ξ_2 may share some edges.) To prove this, let v_1 be any vertex that is matched differently in Ξ_0 and Ξ_1 and v_0 be its match in Ξ_0. Consider the sequence of vertices $v_0, v_1, v_2, v_3, \cdots$ where v_2 is v_1's match in Ξ_0, v_3 is v_2's match in Ξ_1, v_4 is v_3's match in Ξ_0, and so on; the matchings alternate as vertices are added. At some point a repeated vertex $v_j = v_i$ with $j > i$ must appear in the sequence. We now argue that $j \neq i + 2$, so $v_i, v_{i+1}, \ldots, v_{j-1}$ is the desired cycle. In fact, for every $i \geq 0$, v_i and v_{i+2} must be distinct. We can prove this by induction on i: this is true when $i = 0$ by the assumption on v_1, and given that v_i and v_{i+2} are distinct, their matches v_{i+1} and v_{i+3} must also be distinct.

5. G is a directed graph whose vertices are the integers from -10 to 10 (inclusive) and whose edges (x, y) are those ordered pairs for which $|x| - |y| = 1$. For each of the following claims, say if it is true or false and provide a proof.

   (a) G has a path of length 10.

   Solution: True. The path $(10, 9, 8, \ldots, 0)$ has length 10.

(b) G has a parallel schedule of duration 11.

Solution: True. G cannot have a path of length 11 because the absolute value of vertices starts at 10 and must decrease along any path. As G has a parallel schedule whose length is the maximum path length plus one it must have a parallel schedule of size 11.

(c) G has an antichain of size 6.

Solution: True. The set $\{-10, 10, -8, 8, -6, 6\}$ is one such antichain.

6. The vertices of graph H_n are the n integers from $-n$ to n except 0. The edges of H_n are the pairs $\{x, y\}$ such that $x = -y$ or $|y - x| = 1$.

(a) Show that H_n is bipartite.

Solution: Let A be the union of even positive and odd negative vertices and B be the union of even negative and odd positive vertices. There are no edges within A and no edges between B.

(b) How many perfect matchings do H_1 and H_2 have?

Solution: H_1 consists of a single so it has one perfect matching. H_2 is a cycle of length 4 so it has two, namely $\{\{-1, 1\}, \{-2, 2\}\}$ and $\{\{-2, -1\}, \{1, 2\}\}$.

(c) How many perfect matchings does H_{10} have? (**Hint:** Write a recurrence.)

Solution: Let $f(n)$ denote the number of matching of the analogous graph H_n with $2n$ vertices in which the integers -10 and 10 are replaced by $-n$ and n. There are two possible ways in which vertex n can be matched: Either it is matched to $-n$, in which case the remaining vertices to be matched induce the graph H_{n-1}, or it is matched to $n - 1$, in which case $-n$ must also be matched to $-(n + 1)$ and the remaining vertices to be matched induce the graph H_{n-2}. Therefore the number of matchings $f(n)$ satisfies the recurrence $f(n) = f(n - 1) + f(n - 2)$ for all $n \geq 2$ with $f(1) = 1$ and $f(2) = 2$. This is the Fibonacci recurrence and we can calculate the following values for $f(n)$ when $n \leq 10$:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	2	3	5	8	13	21	34	55	89

so $f(10) = 89$.