Practice Midterm 1

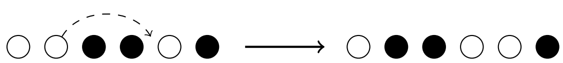
1. Are the propositions "Every two people have a common friend" and "Every person has at least two friends" logically equivalent? Justify your answer.

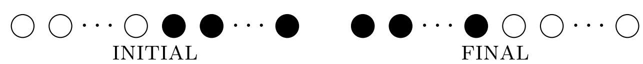
2. Show that for every real number x, at least one of the numbers x, $x + \sqrt{2}$ is irrational.

3. Show that for every integer $n \geq 1$, $1 + 1/4 + 1/9 + \cdots + 1/n^2 \leq 2 - 1/n$.

4. Find the GCD g of 77 and 31 using Euclid's algorithm. Then find a representation of g as an integer linear combination of 77 and 31. Show your work.

5. n white pegs and n black pegs are arranged in a line. In each step you are allowed to move any peg past *two* consecutive pegs of the opposite colour, left or right. Initially all white pegs are to the left of the black ones. Show that the colours can be reversed if and only if n is even.





Practice Midterm 2

1. Is the following deduction rule valid?

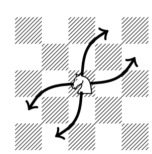
$$\frac{\forall x \exists y \colon P(x,y) \qquad \exists x \forall y \colon P(x,y)}{\forall x \forall y \colon P(x,y)}$$

2. Prove that if $m^2 + n^2$ is even then $m + n$ is even.

3. Alice has an infinite supply of \$4 stamps and exactly three \$7 stamps. Can she obtain all integer postage amounts of \$18 and above? Justify your answer.

4. Show there exists a Die Hard scenario with three jugs and a 1 litre target in which Bruce dies if he can only use any two out of the three jugs to measure, but he survives if he uses all three jugs.

5. A knight jumps around an infinite chessboard. Owing to injury it can only make the moves shown in the diagram. Can it ever reach the square immediately to the left of its initial one?



Practice Midterm 3

1. Underline and explain the mistake in the following "proof."

   Theorem. In every group of friends there exists a person with an even number of friends.

   Proof. By induction on the number of people n. When $n = 1$ the one person has zero friends, and zero is even. Now assume it is true for groups of n people. Let G be a group of $n + 1$ people. Take out any person from G. By inductive hypothesis the remaining group G' has someone, say Alice, with an even number of friends. Since Alice is also in G, G has a person with an even number of friends. $\square$

2. Prove that for every positive integer n, $\gcd(n^2 + n + 1, n + 1) = 1$. (**Hint:** Use the connection between gcd and combinations)

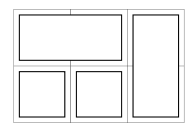
3. Alice has infinitely many \$6, \$10, and \$15 stamps. Can she make all integer postages above \$30?

4. Bob has 32 *b*lue, 33 *r*ed, and 34 *g*reen balls. At every turn he takes out two balls and replaces them with two different balls by the following rule below:

   $$bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb$$

   (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

   (b) Can Bob obtain 99 balls of the same color? Justify your answer. (**Hint:** Consider the difference between the number of red and blue balls.)



5. Use induction to show that for every $n \geq 1$, the $(n + 1) \times n$ grid can be tiled using *two* sets of the following tiles: $1 \times 1, 1 \times 2, \ldots, 1 \times n$. (See example $n = 2$.)