Practice Midterm 1

1. Are the propositions "Every two people have a common friend" and "Every person has at least two friends" logically equivalent? Justify your answer.

   Solution: They are not logically equivalent. Suppose the world consists of Alice, Bob, Charlie, and Dave, and the following friendships: Alice with Bob, Bob with Charlie, Charlie with Dave, Dave with Alice. Then every person has two friends, but Alice and Bob have no common friend.

2. Show that for every real number x, at least one of the numbers x, $x + \sqrt{2}$ is irrational.

   Solution: We prove this by contradiction. Assume that there exists a real number x such that both x and $x+\sqrt{2}$ are rational. The difference of two rational numbers is rational, so $(x+\sqrt{2})-x = \sqrt{2}$ is then rational. This contradicts Theorem 9 from Lecture 2.

3. Find the GCD g of 77 and 31 using Euclid's algorithm. Then find a representation of g as an integer linear combination of 77 and 31. Show your work.

   Solution: Euclid's algorithm operates like this:

$$\begin{aligned} E(77,31) &= E(31,15) & \qquad 77 &= 2\cdot 31 + 15 \\ &= E(15,1) & 31 &= 2\cdot 15 + 1 \\ &= E(1,0) & 15 &= 15\cdot 1 \end{aligned}$$

   so the GCD is 1. The desired combination is

$$1 = 31 - 2\cdot 15 = 31 - 2\cdot(77 - 2\cdot 31) = -2\cdot 77 + 5\cdot 31.$$

4. Show that for every integer $n \geq 1$, $1 + 1/4 + 1/9 + \cdots + 1/n^2 \leq 2 - 1/n$.

   Solution: We prove the proposition by induction on n. In the base case $n = 1$, the left hand side is 1 and the right hand side is $2 - 1/1 = 1$, so the proposition holds. Now take any $n \geq 1$ and assume that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

   Therefore

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$

   We can bound the expression on the right like this:

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{n(n+1)} = 2 - \frac{(n+1)-1}{n(n+1)} = 2 - \frac{1}{n+1}$$

   so the inductive conclusion holds. By induction, the predicate is true for all n.

   A side note: How did I come up with the inequality

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{n(n+1)}?$$

   I did so by working backwards. In order to complete the inductive step, what we *needed* to prove is that

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$

Moving the terms around, I notice that this is equivalent to proving that

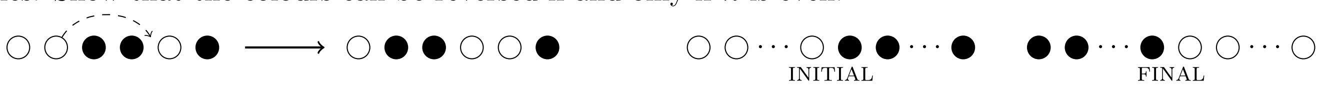
$$\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}.$$

Writing a common denominator for the left hand side, this is the same as

$$\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}.$$

This is certainly true, as the denominator on the right is smaller, so the fraction is larger. This reasoning suggests that a good way to proceed is to bound the term $1/(n+1)^2$ by $1/n(n+1)$. After that we just need to simplify the expression.

5. n white pegs and n black pegs are arranged in a line. In each step you are allowed to move any peg past *two* consecutive pegs of the opposite colour, left or right. Initially all white pegs are to the left of the black ones. Show that the colours can be reversed if and only if n is even.



Solution: First we show that if n is even the colours can be reversed. More generally we show by induction on k that this is true for any number k of white pegs and n black pegs (as long as n is even). When $k = 0$ there are no white pegs so there is nothing to reverse. Now we assume k white pegs and n black pegs can be reversed. Given $k + 1$ white pegs and n black pegs, move the rightmost white peg to the right end by jumping two black pegs at a time and leave it there. By inductive hypothesis the remaining $k + n$ pegs can be reversed, so the whole configuration can be reversed.

Now we show that if n is odd the colours cannot be reversed. Say a pair of pegs is *inverted* if one is black, one is white, and the black one is to the left of the right one. We prove the following invariant: After any number of steps, the number of inverted pairs is even. This is initially true as the number of inverted pairs is zero. Now assume it is true after t steps. In step $t + 1$, the number of inverted pairs goes up by two if a white peg jumps to the right or a black peg jumps to the left, or down by two if a white peg jumps to the left or a black peg jumps to the right. In all cases, the number of inverted pairs stays even.

In the final configuration, every one of the n^2 black-white pairs is inverted. Since n is odd, n^2 is also odd so there is an odd number of inverted pairs. Therefore the final configuration can never be reached.

Practice Midterm 2

1. Is the following deduction rule valid?

$$\frac{\forall x \exists y \colon P(x, y) \qquad \exists x \forall y \colon P(x, y)}{\forall x \forall y \colon P(x, y)}$$

Solution: No. Suppose $P(x, y)$ means "person x is happy on day y", Alice is happy on Monday, Alice is happy on Tuesday, Bob is happy on Monday, but Bob is not happy on Tuesday. Then $\forall x \exists y \colon P(x, y)$ is true because everyone is happy sometimes – on Monday, $\exists x \forall y \colon P(x, y)$ is true because someone (Alice) is happy all the time, but $\forall x \forall y \colon P(x, y)$ is false because Bob is unhappy on Tuesday, so not everyone is happy all the time.

2. Prove that if $m^2 + n^2$ is even then $m + n$ is even.

Solution: We prove the contrapositive: If $m + n$ is odd then $m^2 + n^2$ is odd. Assume $m + n$ is odd. We consider two cases: If m is odd and n is even, then m^2 is odd and n^2 is even, so $m^2 + n^2$ is odd. If m is even and n is odd the same reasoning works with the roles of m and n exchanged. As the two cases cover all possibilities, the statement is true.

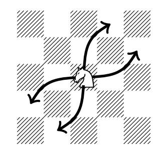
3. Alice has an infinite supply of \$4 stamps and exactly three \$7 stamps. Can she obtain all integer postage amounts of \$18 and above? Justify your answer.

   Solution: Yes. We prove this by strong induction on the postage amount n. When $n = 18$ (the base case), she can obtain \$18 from two \$7 stamps and one \$4 stamp. Now assume this is true for all postage amounts from \$18 up to \$$n$. She can then make \$$(n+1)$ as follows: If $n+1 = 19$, she uses one \$7 and three \$4 stamps. If $n+1 = 20$ she uses four \$5 stamps. If $n+1 = 21$ she uses three \$7 stamps. If $n+1 \geq 22$, then $n-3 \geq 18$ so by inductive assumption she can make \$$(n-3)$ using \$4 stamps and at most three \$7 stamps. Using an additional \$4 stamp she obtains \$$(n+1)$. It follows that she can obtain any amount above \$18 by strong induction on n.

4. Show there exists a Die Hard scenario with three jugs and a 1 litre target in which Bruce dies if he can only use any two out of the three jugs to measure, but he survives if he uses all three jugs.

   Solution: Suppose Bruce has a 6 litre, a 10 litre, and a 15 litre jug. Since 2 divides both 6 and 10, Bruce cannot measure 1 litre using these two jugs. Since 3 divides both 6 and 15, Bruce also cannot measure 1 litre with them. Since 5 divides both 10 and 15, Bruce dies if he restricts himself to using those two only. However, if all three jugs are available, Bruce can measure 1 litre by filling the 6 and 10 litre jugs to the top, then pouring out their contents into the 15 litre jug until it fills up. There will be 1 litre left in one of them.

5. A knight jumps around an infinite chessboard. Owing to injury it can only make the moves shown in the diagram. Can it ever reach the square immediately to the left of its initial one?



   Solution: No. We represent each chessboard square by its integer (x, y) coordinates, with the initial square being $(0, 0)$ and the one to its left $(-1, 0)$. If we model the jumping knight by a state machine, the transitions out of state (x, y) are

$$(x,y) \to (x-2, y-1) \quad (x,y) \to (x-1, y-2) \quad (x,y) \to (x+2, y+1) \quad (x,y) \to (x+1, y+2).$$

   The predicate "3 divides $x + y$" is an invariant of this state machine. It holds initially, and after every transition $(x, y) \to (x', y')$ we have $x' + y' = x + y - 3$ in the first two cases and $x' + y' = x + y + 3$ in the other two. Assuming the invariant holds before the transition (i.e., 3 divides $x + y$) it also holds after the transition (3 divides $x' + y'$).

   The invariant does not hold for state $(-1, 0)$ so that state cannot be reached.

Practice Midterm 3

1. Underline and explain the mistake in the following "proof."

   Theorem. In every group of friends there exists a person with an even number of friends.

   Proof. By induction on the number of people n. When $n = 1$ the one person has zero friends, and zero is even. Now assume it is true for groups of n people. Let G be a group of $n + 1$ people. Take out any person from G. By inductive hypothesis the remaining group G' has someone, say Alice, with an even number of friends. <u>Since Alice is also in G, G has a person with an even number of friends.</u> □

   Solution: If Alice has an even number of friends in G' we cannot conclude she has an even number of friends in G'. Her number of friends in G and G' may be of different parity. For example if G consists of Alice and Bob and they are friends then Alice has an odd number of friends in G but after removing Bob to obtain G', Alice is left alone and has zero (an even number) of friends.

2. Prove that for every positive integer n, $\gcd(n^2 + n + 1, n + 1) = 1$. (**Hint:** Use the connection between gcd and combinations)

   Solution: 1 is an integer combination of the two numbers: $1 = 1 \cdot (n^2 + n + 1) - n \cdot (n + 1)$. As the GCD must divide all integer combinations it must equal one.

3. Alice has infinitely many \$6, \$10, and \$15 stamps. Can she make all integer postages above \$30?

   Solution: Alice can make all integer postages from \$30 to \$35 as follows:

$$
\begin{aligned}
\$30 &= 5 \times \$6 \\
\$31 &= \$6 + \$10 + \$15 \\
\$32 &= 2 \times \$6 + 2 \times \$10 \\
\$33 &= 3 \times \$6 + \$15 \\
\$34 &= 4 \times \$6 + \$10 \\
\$35 &= 2 \times \$10 + \$15
\end{aligned}
$$

   Now we show that she can make any amount n above 30 by strong induction on n. We already covered the cases $30 \leq n \leq 35$. Now assume that $n > 35$ and she can make all amounts between \$30 and \$$n$. Then $n - 6 \geq 30$ and by inductive assumption she can make $n - 6$ dollars. By adding one \$6 stamp she obtains n dollars.

4. Bob has 32 *b*lue, 33 *r*ed, and 34 *g*reen balls. At every turn he takes out two balls and replaces them with two different balls by the rule below. Can he obtain 99 balls all of the same color?

   replacement rule: $bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb$

   (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

   Solution: the states are triples (B, R, G) indicating the number of balls of each color. The start state is $(32, 33, 34)$. The transitions are from (B, R, G) to the states $(B-1, R-1, G+2)$, $(B+2, R-1, G-1)$, $(B-1, R-1, G+2)$, $(B+1, R-2, G+1)$, $(B-2, R+1, G+1)$, $(B+1, R+1, G-2)$ as long as all numbers remain non-negative.

   (b) Can Bob obtain 99 balls of the same color? Justify your answer. (**Hint:** Consider the difference between the number of red and blue balls.)

   Solution: The predicate "3 does not divide $R - B$" is an invariant. It holds in the start state and it is preserved by all transitions as $R - B$ can only change by -3, 0, or 3. If all 99 balls are of the same color then 3 divides $R - B$, so that state cannot be reached. (Another useful invariant is "$R - B$ is of the form $3k + 1$.")

5. Use induction to show that for every $n \geq 1$, the $(n + 1) \times n$ grid can be tiled using *two* sets of the following tiles: $1 \times 1, 1 \times 2, \ldots, 1 \times n$.

   Solution: In the base case $n = 1$, we tile the 2×1 grid by putting two 1×1 tiles side by side. For the inductive step, assume that the $(n + 1) \times n$ grid (where $n \geq 1$) can be tiled using two sets of the tiles 1×1 up to $1 \times n$. We show that the $(n + 2) \times (n + 1)$ grid can be tiled using two sets of the tiles 1×1 up to $1 \times (n + 1)$: Take the tiling of the $(n + 1) \times n$ grid, add an $(n + 1) \times 1$ horizontal tile to the top of it and a $1 \times (n + 1)$ vertical tile to the right of it. We obtain the desired tiling of the $(n + 2) \times (n + 1)$ grid.

   By induction, it follows that the proposition is true for all $n \geq 1$.