

1. (9 points) Let M be the matrix

$$M = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}.$$

- (a) 0 is an eigenvalue of M .

Solution: True. The first three rows of M are linearly dependent. They are all in the span of $(0, 1, 2, 3)$ and $(1, 1, 1, 1)$. M does not have full rank, so the system $Mx = 0$ has a solution and zero is an eigenvalue.

- (b) i (the imaginary unit) is an eigenvalue of M .

Solution: False. All eigenvalues of real symmetric matrices are real.

- (c) -100 is an eigenvalue of M .

Solution: False. -100 is of too large magnitude to be a possible eigenvalue. If the largest entry (in absolute value) in the corresponding eigenvector x was b , then the entries of Mx would be at most $3b + 4b + 5b + 6b = 18b$. This puts an upper bound of 18 on the eigenvalues.

The reason is not that -100 is negative. Matrices with positive entries can have negative eigenvalues.

2. (9 points) Let $X_1, X_2, X_3 \in \{0, 1\}$ be independent uniform random bits. Let N be the random variable

$$N = \begin{cases} 1, & \text{if } X_1 = X_2 = X_3 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) X_1 and N are identically distributed.

Solution: False. $P(N = 1) = 1/4$, while $P(X_1 = 1) = 1/2$.

- (b) X_1 and N are independent.

Solution: True. $P(X_1 = 1 \text{ and } N = 1) = P(X_1 X_2 X_3 = 111) = 1/8$. This is the product of $P(X_1 = 1)$ and $P(N = 1)$.

- (c) X_1, X_2 , and N are independent.

Solution: False. $P(X_1 = 0 \text{ and } X_2 = 1 \text{ and } N = 1)$ is zero, but the product of the marginal probabilities is $1/2 \cdot 1/2 \cdot 1/4 \neq 0$.

3. (8 points) Let A be the following algorithm:

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1  Input: A positive integer  $n$ .
2  While  $\sqrt{n}$  is an integer,
3      Replace  $n$  with  $\sqrt{n}$ .
4  Output  $n$ .
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- (a) Fill in the blank: $A(n)$ outputs 2 if and only if n equals 2^{2^t} for some nonnegative integer t .

Solution: Output 2 was produced if n had value 2^2 in the last step of the iteration, or value $(2^2)^2 = 2^{2^2}$ in the previous step, or value $2^{2^{2^2}} = 2^{2^3}$ in the previous one, and so on.

- (b) Let $t(n)$ be the number of times loop 2-3 is executed on input n . Give the best upper bound you can for $t(n)$. You may use big-oh notation.

Solution: $\log \log n$ (logarithms are base two). If output m was produced after t steps of iteration then $n = m^{2^t}$. As m must be at least two, $n \geq 2^{2^t}$. Taking logs twice we get $t \leq \log \log n$. (The bound is tight by part (a).)