

Exam 1

1. Systems A and B consist of four linear equations in four unknowns modulo 2.

$$\begin{array}{ccccccc} x_1 & + & x_2 & = & 0 & & \\ + & & + & & & & \\ y_1 & + & y_2 & = & 0 & & \\ = & & = & & & & \\ 1 & & 1 & & & & \end{array}$$

system A

$$\begin{array}{ccccccc} x_1 & + & x_2 & = & 1 & & \\ + & & + & & & & \\ y_1 & + & y_2 & = & 0 & & \\ = & & = & & & & \\ 1 & & 1 & & & & \end{array}$$

system B

- Solve system A using modular Gauss-Jordan elimination. If there are multiple solutions, specify the free variables and the assignment to the remaining variables in terms of the free variables.
- In system B the value in the first equation was flipped. Produce a contradictory linear combination for system B . You may use any method you like.

2. You want to apply Gradient Descent to the system

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = 4$ and $\lambda_2 = -2$ with corresponding eigenvectors $\mathbf{v}_1 = (1/\sqrt{2}, 1/\sqrt{2})$, $\mathbf{v}_2 = (1/\sqrt{2}, -1/\sqrt{2})$.

- What is the maximum rate ρ^* below which convergence to the unique solution $(1/4, 1/4)$ is guaranteed (for any initialisation)?
- You run Gradient Descent with rate $\rho = 0.01$ and initialisation $(x, y) = (1, 0)$. The distance between the state (x_t, y_t) at time t and the solution $(1/4, 1/4)$ is $\Theta(b^t)$ for some number b . Find b .

3. Calculate the Fourier transforms of the following functions. You may use any method you like. $+$ and $-$ are shorthand for the numbers $+1$ and -1 .

$$(a) \frac{x_1 x_2}{f(x_1 x_2)} \begin{array}{c|cccc} & ++ & -+ & +- & -- \\ \hline & 1 & 1 & -1 & -1 \end{array}$$

$$(b) \frac{x_1 x_2}{g(x_1 x_2)} \begin{array}{c|cccc} & ++ & -+ & +- & -- \\ \hline & 0 & 1 & 1 & 1 \end{array}$$

(Hint: What is $1 - g$?)

$$(c) h(x_1 x_2 x_3) = \begin{cases} f(x_1 x_2), & \text{if } x_3 = -1 \\ g(x_1 x_2), & \text{if } x_3 = +1. \end{cases}$$

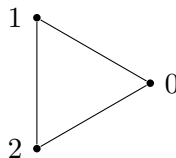
(Hint: Apply one of the steps of the Fast Fourier-Walsh algorithm.)

Exam 2

1. Let S be the system of equations

$$\begin{aligned}x + y + z &= 0 \\x + 2y + 4z &= 0\end{aligned}$$

- (a) Apply Gauss-Jordan elimination to find a nonzero solution. Explain your steps.
- (b) Now add the equation $x + 3y + 5z = 1$ to the system. Find a contradictory linear combination.
2. Let A be the adjacency matrix of the 3-cycle (with zeros on the diagonal). The largest eigenvalue of A is $\lambda_1 = 2$. The corresponding eigenvector is $\mathbf{v}_1 = (1/\sqrt{3})(1, 1, 1)$.



- (a) Let $\mathbf{x}$ be any vector that is orthogonal to $\mathbf{v}_1$. Show that the state of Power Iteration on A initialized with $\mathbf{x}$ oscillates between $\mathbf{x}$ and $-\mathbf{x}$. Assume there are no precision errors.
- (b) Use part (a) to deduce the other two eigenvalues λ_2 and λ_3 of A . What are they?
3. Apply the modular Fast Fourier transform to calculate the polynomial representation of

$$\begin{array}{c|cccc}x & 1 & i & -1 & -i \\ \hline f(x) & 0 & 1 & 2 & 3\end{array}$$

- (a) Calculate the list representations of f^+ and f^- , where $f(x) = f^+(x^2) + xf^-(x^2)$.
- (b) Calculate the polynomial representations of f^+ and f^- over domain $\{-1, +1\}$.
- (c) Calculate the polynomial representation of f .

1 Exam 3

1. You apply gradient descent to the following (inconsistent) univariate linear system:

$$x = 0$$

$$x = 1$$

- (a) Write the sum-of-squares loss f . For which value of $x^\star$ is $f(x^\star)$ minimized?
- (b) Let x_t be the state after t steps of gradient descent with rate ρ . What is $(x_t - x^\star)/(x_{t-1} - x^\star)$?
- (c) Assuming $x_0 = 0$ and $\rho = 1/8$, what is x_{20} ?

2. You apply subspace iteration on the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Your initial state is the standard basis $\mathbf{x}_1 = (1, 0, 0)$, $\mathbf{x}_2 = (0, 1, 0)$, $\mathbf{x}_3 = (0, 0, 1)$.

- (a) Ignoring $\mathbf{x}_3$ for now, show how $\mathbf{x}_1$ and $\mathbf{x}_2$ evolve in the first three rounds of subspace iteration.
 - (b) Use part (b) to find two of the three eigenvalues of A . Explain your reasoning.
 - (c) Prove that if, at any point during power iteration, $A\mathbf{x}_i$ and $A\mathbf{x}_j$ become linearly dependent (for some $i \neq j$), zero must be an eigenvalue of A .
 - (d) Use part (c) to find the remaining eigenvalue of A . (**Hint:** What happens to $\mathbf{x}_3$?)
3. The *2-bit addressing function* $\text{addr}: \{+1, -1\}^6 \rightarrow \{+1, -1\}$ is given by

$$\text{addr}(x, y, z_{++}, z_{+-}, z_{-+}, z_{--}) = z_{xy}.$$

- (a) Express addr in terms of the four point functions $\text{point}_{ab}(x, y)$ and the four variables z_{ab} .
- (b) Calculate $\widehat{\text{addr}}(\emptyset)$, $\widehat{\text{addr}}(\{1, 2, 3\})$ and $\widehat{\text{addr}}(\{1, 2, 3, 4\})$. (Use part (a).)