You are encouraged to collaborate on the homework and ask for assistance. You are required to write your own solutions, list your collaborators, acknowledge all sources of help, and cite all external references.

Question 1

The intersection function $INT\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is $INT(x,y) = (x_1 \text{ AND } y_1) \text{ OR } \cdots \text{ OR } (x_n \text{ AND } y_n)$. Show that

(a) INT requires a (deterministic) read-once branching program of width 2^n (in the order $x_1, \ldots, x_n, y_1, \ldots, y_n$).

(b) $r_0(INT) \leq 2^n$, where $r_0(f)$ is the size $|X| \cdot |Y|$ of the largest rectangle $X \times Y$ for which $f(x,y) = 0$ for all $x \in X$ and $y \in Y$. (**Hint:** Reduce to IP)

(c) If $f(x,y)$ can be computed by a width-w read-k-times branching program then f can evaluate to zero on at most $r_0(f)w^{2k}$ inputs.

(d) Use parts (c) and (d) to show that INT requires read-k-times branching program width at least $(3/2)^{n/2k}$.

Question 2

Let X be an n by n matrix and $f\colon \{0,1\}^{n^2} \to \{0,1\}$ be the function

$$f(X) = \begin{cases} 1, & \text{if } f \text{ has } \textit{exactly one} \text{ column consisting of zeros only,} \\ 0, & \text{otherwise.} \end{cases}$$

Determine the following quantities up to a constant factor (i.e., in $\Theta(\cdot)$ notation). Provide both upper and lower bound proofs.

(a) the deterministic query complexity $D(f)$

(b) the exact degree $\deg(f)$ when f is viewed as a real-valued polynomial

(c) the sensitivity $\mathrm{sens}(f)$

(d) **(Optional)** the Monte Carlo randomized query complexity $R(f)$

(e) **(Optional; possible project)** the quantum query complexity $Q(f)$

Question 3

The *correlation* between two strings $a, b \in \{-1, 1\}^n$ is the number $\langle a, b\rangle/n = (a_1b_1 + \cdots + a_nb_n)/n$ in the range $[-1, 1]$. You will study the classical and quantum query complexities of estimating correlation. An *unbiased estimator* for correlation is an algorithm that accepts (x_0, x_1) with probability $\frac{1}{2} + \frac{1}{2}\langle x_0, x_1\rangle/n$. The input $x = (x_0, x_1)$ is represented as the $2n$-bit string $x_{01} \cdots x_{0n}x_{11} \cdots x_{1n}$. Show that

(a) There exists a 2-query randomized unbiased estimator for correlation.

(b) Any 1-query randomized algorithm has the same acceptance probability on the input distributions

$$\{(X_0, X_1) : X_0 \text{ and } X_1 \text{ are the same random } n\text{-bit string}\} \quad \text{and}$$
$$\{(X_0, X_1) : X_0, X_1 \text{ are independent random } n\text{-bit strings}\}.$$

(**Hint:** Argue this for deterministic algorithms first.)

(c) There does not exist a 1-query randomized unbiased estimator for correlation.
(**Hint:** Can the algortihm answer correctly in expectation on both distributions in part (b)?)

(d) The quantum algorithm

Measure the first qubit of $H_1\Phi^x|+\rangle$ and accept if it is zero

is a (1-query) unbiased estimator for correlation. Here, $|+\rangle$ is the state $(|01\rangle + \cdots + |0n\rangle + |11\rangle + \cdots + |1n\rangle)/\sqrt{2n}$ and H_1 is the Hadamard gate applied to the first qubit $|b\rangle$. In ± 1 bit representation Φ^x is the phased-query gate $\Phi^x|bi\rangle = x_{bi}|bi\rangle$.

(e) **(Optional)** There is a 1-query quantum unbiased estimator of $\frac{1}{n}\sum A_{ij}x_{0i}x_{1j}$ for every $n \times n$ orthogonal (real unitary) matrix A.