CSI 5138: Computational Complexity — Midterm Exam
University of Ottawa, Fall 2023

Please submit your solution here by Wednesday 25 October. No collaboration is allowed on this assignment.

Question 1

For each of the following claims about the ‘inner product mod 2” function $IP\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

$$IP(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \bmod 2.$$

say if it is true or false (for all sufficiently large n). Prove your assertion (referring to results from the lecture notes if needed).

(a) IP has a width-4 read-once branching program when the input is read in order $x_1, y_1, \ldots, x_n, y_n$.

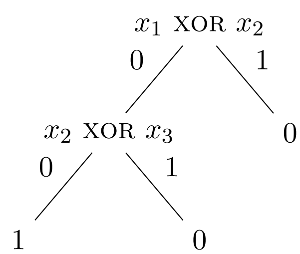
(b) IP has a width-$(2^n - 1)$ read-once branching program when the input is read in order $x_1, \ldots, x_n, y_1, \ldots y_n$.

(c) IP has deterministic query complexity $2n$.

(d) IP has a depth-4 AND/OR circuit of size at most n^2.

Question 2

A *parity decision tree (PDT)* is a generalized type of decision tree that can query parities of arbitrary subsets of the variables. For example this is a depth-2 PDT for the function “x_1, x_2, and x_3 are all equal”.



Show that

(a) Every depth-d PDT can be computed by a $\mathbb{F}_2$-polynomial ($+$ is XOR, $\times$ is AND) of degree at most d.
(**Hint:** Modify the proof of Claim 4 from Lecture 4.)

(b) AND of n inputs requires PDT depth n. (Use part (a).)

(c) MAJORITY on n inputs requires PDT size $\Omega(2^{n^\epsilon})$ for some constant $\epsilon > 0$.
(**Hint:** Convert the PDT to an AND/OR/PARITY circuit.)

Question 3

Let $BPSAT$ be the decision problem whose input is a branching program B and whose YES instances are those B that accept at least one of their inputs. Let $ROBPSAT$ be the analogous decision problem for *read-once* branching programs. Argue that

(a) $BPSAT$ and $ROBPSAT$ are in NP.

(b) $ROBPSAT$ is in P.

(c) $BPSAT$ is NP-complete.
(**Hint:** Reduce from SAT to $BPSAT$.)