ENGG 2440A: Discrete Mathematics for Engineers — Homework 4
The Chinese University of Hong Kong, Fall 2014 — due Friday 7 November

Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

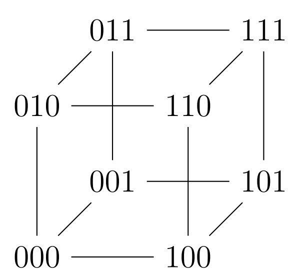
Questions

1. Prove the following propositions. For part (b), the *in-degree* and *out-degree* of a vertex in a digraph is its number of incoming and outgoing edges, respectively.

   (a) Every graph has an even number of vertices of odd degree.

   (b) In every digraph, the sum of the in-degrees of its vertices equals the sum of the out-degrees of its vertices.

2. The *hypercube* H_n of dimension n is the following graph on 2^n vertices: The vertices of H_n are all $\{0, 1\}$ strings of length n. There is an edge for any two vertices that differ in exactly one bit position. Here is a diagram of H_3:



   (a) Show that for every $n \geq 1$ and every two vertices u, v in H_n there is a path from u to v of length at most n.

   (b) Show that for every $n \geq 1$, H_n is a bipartite graph.

3. You have a bipartite graph where the vertices are partitioned into 10 boys and 20 girls. Every boy vertex has degree 6. Every girl vertex has degree 3. Show that there exists a matching that matches all the boys.

4. Say vertex u is *reachable* from vertex v in at most ℓ steps if there exists a path from u to v of length at most ℓ. You are given a connected graph G with n vertices in which the degree of every vertex is at most d, where $d \geq 2$.

   (a) Let v be a vertex in G. Use induction to prove that for every $\ell \geq 0$, the number of vertices that are reachable from v in at most ℓ steps is strictly less than $d^{\ell+1}$.

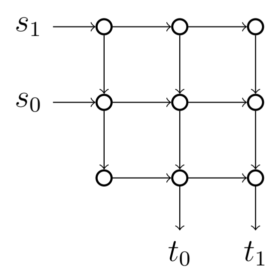
(b) Use part (a) to show that there exists two vertices u, v in G such that every path from u to v has length at least $\lfloor \log_d n \rfloor$. ($\lfloor x \rfloor$ is the largest integer not larger than x, e.g., $\lfloor 4.21 \rfloor = \lfloor 4 \rfloor = \lfloor 4.7 \rfloor = 4$.)

(c) **(Extra credit)** Show that if every vertex in G has degree exactly d, then for every pair of vertices u, v in G, there is a path from u to v of length at most $3n/d$.

5. Let $m, n \geq 3$ be two integers. Consider the following directed graph G:

- The vertices of G consist of the grid points (i, j), $1 \leq i \leq m, 1 \leq j \leq n$ together with the four special elements s_0, t_0, s_1, t_1.
- The edges of G are of the form $((i, j), (i + 1, j))$ for every $1 \leq i < m$ and $1 \leq j < n$ and $((i, j), (i, j - 1))$ for every $1 \leq i \leq m$ and $1 < j \leq n$ together with the four edges $(s_0, (1, 2)), (s_1, (1, n)), ((2, 1), t_0), ((m, 1), t_1)$.

Here is a diagram of G when $m = n = 3$:



(a) Show that for all $m, n \geq 3$, G has no cycles. (One way to show this is to give a topological sort of G with a proof that it is a topological sort.)

(b) Show that for all $m, n \geq 3$, G is not a switching network.

6. True or false: For some $n \geq 3$ there exists a set of n boys, n girls, and preference lists for every boy and girl such that *every* possible boy-girl matching is stable. If true, give a proof. If false, give a refutation (prove the negation of the statement).