Each question is worth 10 points. Please explain your solution clearly and concisely.

1. Show that $\sqrt{2} + \sqrt{3}$ is an irrational number.

   Solution: For contradiction, assume $\sqrt{2} + \sqrt{3}$ is rational. Since the square of a rational number is also rational, $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ is rational too. If we take a rational number, subtract 5 and divide by 2 we also get a rational number, so $\sqrt{6}$ must be rational.

   We can therefore write $\sqrt{6} = m/n$, where m and n are integers with no common factors. After squaring both sides and rearranging terms we get $m^2 = 6n^2$, so m^2 is even, therefore m is also even. If we write $m = 2m'$ we get $4m'^2 = 6n^2$, so $2m'^2 = 3n^2$. It follows that n^2 must be even, and n also. This contradicts the assumption that m and n have no common factors.

2. You have a 6 litre, a 10 litre, and a 45 litre jug, and a water source. Can you measure 1 litre using the pouring rules from Lecture 4?

   Solution: Yes you can. In lecture 4 we showed that using the 10 litre and 45 litre jug you can measure $\gcd(10, 45) = 5$ litres. Measure these 5 litres and pour them into the 10 litre jug. Fill the 6 litre jug to the top, pour its contents into the 10 litre jug until that one fills up, and you are left with exactly 1 litre in the 6 litre jug.

3. Show that in every graph, the sum of the squares of the degrees of all the vertices is an even number.

   Solution: In Lecture 5 we showed that the sum of the degrees of all the vertices is an even number. If there are n vertices and their degrees are $d_1, d_2, \dots, d_n$, then $d_1 + \cdots + d_n = 2k$ for some integer k. Squaring both sides we get

   $$d_1^2 + \cdots + d_k^2 + 2d_1d_2 + 2d_1d_3 + \cdots + 2d_{n-1}d_n = 4k^2$$

   so $d_1^2 + \cdots + d_k^2$ is the difference of two even numbers, and therefore even.

4. Show that in every digraph in which there is no source there are two vertices of the same in-degree.

   Solution: Let n be the number of vertices. The in-degree of every vertex is a in integer between 0 and $n-1$. Since there is no source, there is no vertex of in-degree zero. So there are $n-1$ possible values for the in-degree and n vertices. By the pigeonhole principle, two vertices must have the same in-degree.

5. Show that $1 + 1/4^2 + 1/7^2 + \cdots + 1/(3n+1)^2$ is $\Theta(1)$.

   Solution: Clearly $1 + 1/4^2 + 1/7^2 + \cdots + 1/(3n+1)^2 \geq 1$, so it is $\Omega(1)$. One way to show that the sum is $O(1)$ is to upper bound it by

   $$1 + 1/4^2 + 1/7^2 + \cdots + 1/(3n+1)^2 \leq 1 + 1/2^2 + 1/3^2 + \dots$$

   This is a decreasing sequence, so we can use the integral method to upper bound it by

   $$1 + 1/2^2 + 1/3^2 + \cdots \leq 1 + \int_1^\infty \frac{1}{x^2} dx = 1 - \frac{1}{x}\Big|_1^\infty = 2.$$

6. How many five-card poker hands are there that have the same number of kings and aces?

   Solution: Let A_0, A_1, and A_2 be the set of poker hands with zero kings and zero aces, one king and one ace, and two kings and two aces, respectively. Since these sets are disjoint, by the sum rule our answer is the number $|A_0| + |A_1| + |A_2|$.

   We find the sizes of these sets. The set A_0 consists of all five-card hands chosen from the 44 cards that exclude the kings and aces, so $|A_0| = \binom{44}{5}$. We count $|A_1|$ and $|A_2|$ using the product rule. For A_1, we can choose the king in four ways, the ace in four ways, and the other three cards in $\binom{44}{3}$ ways, so $|A_1| = 4^2 \cdot \binom{44}{3}$. For A_2, there are $\binom{4}{2}$ choices for the pair of kings, another $\binom{4}{2}$ for the pair of aces, and 44 for the last card. So the desired count is

   $$\binom{44}{5} + 4^2 \cdot \binom{44}{3} + \binom{4}{2}^2 \cdot 44.$$

   (It is okay to leave the answer in this form.)

7. Let p be a polynomial of the form $p(x) = ax^4 + bx^3 + x^2$ over $\mathbb{F}_q$, where q is a prime number. Show that p has at most three zeros.

   Solution: We can write $p(x) = x^2(ax^2 + bx + 1)$. If $p(x) = 0$, it must be that $x = 0$ or if $x \neq 0$ then $ax^2 + bx + 1 = 0$. In Lecture 11 we showed that a degree two polynomial like $ax^2 + bx + 1$ has at most two zeros. So p has at most three zeros.

8. Alice comes up with a circular seating arrangement for n guests at a round table. Show that Bob needs to ask Alice $\Omega(n \log n)$ yes/no questions in order to figure out the arrangement with certainty.

   Solution: In Lecture 9 we showed that if Alice thinks up one of N "secrets", the number of yes/no questions Bob needs to ask Alice in order to figure out her secret with certainty is $\log N$. The number of circular arrangements for n guests is $N = (n-1)!$, so to figure out the arrangement Bob needs to ask

   $$\log(n-1)! = \log n! - \log n = \Omega(n \log n) - \log n = \Omega(n \log n)$$

   questions.