Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Questions

1. In which of the following Die Hard scenarios does Bruce survive? Justify your answer.

   (a) Measure 11 litres of water with a 7,469 litre and a 2,464 litre jug.

   (b) Measure 4 litres of water with an 18 litre, a 24 litre, and a 33 litre jug.

   (c) Measure 1 litre of water with a 6 litre, a 10 litre, and a 15 litre jug.

2. Does there exist a graph of the following type? If a graph exists, describe it. If not, prove that it doesn't.

   (a) A graph with 9 vertices in which every vertex has degree 4.

   (b) A graph with 9 vertices in which every vertex has degree 3.

   (c) A graph with 9 vertices in which the sum of the degrees of all the vertices is 40.

3. Solve the following (systems of) equations in modular arithmetic. Justify your steps.

   (a) $1915x = 204 \pmod{9851}$ (9851 is a prime number).

   (b) $4x + 7y = 2 \pmod{11}$ and $8x + 9y = 8 \pmod{11}$.

   (c) **(Extra credit)** $x^2 = 5969 \pmod{9851}$. (I recommend you use a computer for this one.)

4. Prove the following statements.

   (a) For all $n \geq 5$, n is composite if and only if n divides $(n-1)!$.

   (b) For all $n \geq 1$, $\gcd(21n + 4, 14n + 3) = 1$.

   (c) (**Extra Credit**) There exists a positive integer n such that $n/3$ is a third power, $n/5$ is a fifth power, and $n/7$ is a seventh power of an integer. (Do not just write down a number; explain how you discovered it.)

5. The greatest common divisor of three integers a, b, c is the largest integer that divides all three of them. Consider the following algorithm A:

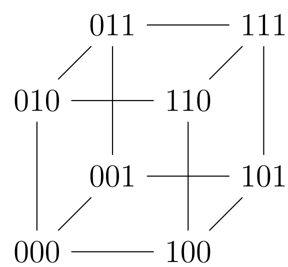
   Algorithm $A(a, b, c)$, where a, b, c are non-negative integers:
   Reorder the three integers (if necessary) so that $a \geq b \geq c$.
   If $c = 0$, run Euclid's algorithm on input (a, b) and output its answer.
   Otherwise, return $A(a - c, b - c, c)$.

(a) Apply A to calculate the gcd of 6, 10, and 15. Show the steps that the algorithm takes.

(b) Prove that for all non-negative integers a, b, c, $A(a, b, c)$ terminates and outputs $\gcd(a, b, c)$.

(c) Find integers u, v, w such that $6u + 10v + 15w = \gcd(6, 10, 15)$. Show and explain your work.

6. The *hypercube* H_n of dimension n is the following graph on 2^n vertices: The vertices of H_n are all $\{0, 1\}$ strings of length n. There is an edge for any two vertices that differ in exactly one bit position. Here is a diagram of H_3:



(a) Show that for every $n \geq 1$, H_n is a bipartite graph.

(b) Show that for every $n \geq 1$, H_n has a perfect matching.

(c) Now assume that n is odd and let G_n be the graph obtained by removing all vertices from H_n except those that have exactly $(n-1)/2$ zeroes or $(n-1)/2$ ones. Give perfect matchings for the graphs G_3 and G_5.

(d) Prove that for every odd $n \geq 1$, G_n has a perfect matching.