

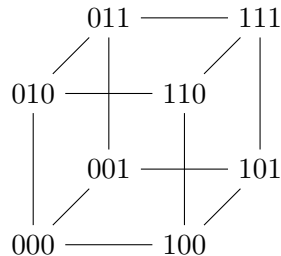
Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without credit will be considered plagiarism and may result in failing the whole course.

### Questions

- In which of the following Die Hard scenarios does Bruce survive? Justify your answer.
  - Measure 14 litres of water with a 203 litre and a 259 litre jug.
  - Measure 5 litres of water with an 16 litre, a 24 litre, and a 32 litre jug.
  - Measure 1 litre of water with a 7 litre and a 9 litre jug in at most 12 pouring steps.
- Do the following graphs exist? If yes, give an example. If not, prove that it isn't possible.
  - A graph with 7 vertices of degrees 1, 2, 3, 4, 3, 2, 1, respectively.
  - A graph with 9 vertices of degrees 1, 2, 3, 4, 5, 4, 3, 2, 1, respectively.
  - A graph with 100 vertices, in which 10 vertices have degree 90 each and each of the remaining 90 vertices has degree 8.
- Solve the following (systems of) equations in modular arithmetic. Justify your steps.
  - $6247x \equiv 1139 \pmod{9461}$ . (9461 is a prime number.)
  - $7x + 3y \equiv 9 \pmod{17}$  and  $16x + 14y \equiv 3 \pmod{17}$ .
  - (Extra credit)**  $x^3 \equiv 123\,456\,789 \pmod{1\,000\,000\,007}$ . (The modulus is a prime.)
- Which of the following graphs  $G = (V, E)$  has a perfect matching? If a graph has a perfect matching, describe it. Otherwise, prove that a perfect matching does not exist.
  - The vertices are all integers between 1 and 1000. The edges are those  $\{x, y\}$  such that  $\gcd(x, y) = 1$ .
  - The vertices are all composite numbers between 1 and 100. The edges are those  $\{x, y\}$  such that  $\gcd(x, y) > 1$ .
  - The vertices are all integers between  $-100$  and  $100$  except for 0. The edges are those  $\{x, y\}$  such that  $0 > x \cdot y > -2345$ .
- The greatest common divisor  $\gcd(a, b, c)$  of three integers  $a, b, c$  is the largest integer that divides all three of them.
  - Prove that for all  $a, b$ , and  $c$ ,  $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$ .  
**(Hint:** Show that each number is no larger than the other.)
  - Prove that if  $a, b, c$  are not all zero then  $\gcd(a, b, c)$  is an integer combination of  $a, b$ , and  $c$ . Namely, there exist integers  $s, t$ , and  $u$  such that  $\gcd(a, b, c) = s \cdot a + t \cdot b + u \cdot c$ .  
**(Hint:** Use part (a) and results from class.)

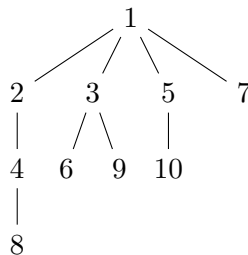
- (c) Find integers  $s, t, u$  such that  $1479s + 2397t + 1363u = 1$ . Show and explain your work. You may use the code provided in class.
6. The *hypercube*  $H_n$  of dimension  $n$  is the following graph on  $2^n$  vertices: The vertices of  $H_n$  are all  $\{0, 1\}$  strings of length  $n$ . There is an edge for any two vertices that differ in exactly one bit position. Here is a diagram of  $H_3$ :



- (a) Show that for every  $n \geq 1$ ,  $H_n$  is a bipartite graph.
- (b) Show that for every  $n \geq 1$ ,  $H_n$  has a perfect matching.
- (c) Now assume that  $n$  is odd and let  $G_n$  be the graph obtained by removing all vertices from  $H_n$  except those that have exactly  $(n-1)/2$  zeroes or  $(n-1)/2$  ones. Give perfect matchings for the graphs  $G_3$  and  $G_5$ .
- (d) Prove that for every odd  $n \geq 1$ ,  $G_n$  has a perfect matching.

**ESTR 2004 mini-project** In this project you will seek the smallest vertex cover that you can find for the following graphs. You may use any of the algorithms described in Lecture 5, an algorithm that you devise, mathematical reasoning, or any mix of methods that you find suitable.

- (a) The hypercube  $H_8$  described in question 6.
- (b) The vertices are the integers from 1 to 100, and  $\{u, v\}$  is an edge if and only if  $u$  is that largest number less than  $v$  that divides  $v$ . This is the part induced by vertices 1 to 10:



- (c) The vertices are four-symbol strings  $x_1x_2x_3x_4$ , where each  $x_i$  is 0, 1, 2, 3, or 4. The pair  $\{x_1x_2x_3x_4, y_1y_2y_3y_4\}$  is an edge if and only if for every  $i$ ,  $x_i - y_i$  is 0, 1, or 4 modulo 5.

Describe your solution clearly. Explain how it compares to the best possible. If you used a computer program to calculate the vertex cover, please also submit your solution in a file by e-mail to [engg2440a@gmail.com](mailto:engg2440a@gmail.com). There should be a separate file for each part containing a single line representing the list of the vertices in the cover separated by spaces.