

Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without credit will be considered plagiarism and may result in failing the whole course.

Questions

- Find exact closed form expressions for the following sums. Explain how you discovered the expression and prove that it is correct.
 - $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n + 1)$.
 - $1^2 + 3^2 + 5^2 + \cdots + n^2$, where n is odd.
 - $1 \cdot 2^{-1} + 2 \cdot 2^{-2} + \cdots + n \cdot 2^{-n}$. (**Hint:** Calculate $2S - S$, where S is the sum of interest.)
- Show the following inequalities by using the integral method for approximating sums.
 - $2\sqrt{n+1} - 2 \leq 1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} \leq 2\sqrt{n+1} - 1$.
 - $\frac{1}{2}n^2 \ln n - \frac{1}{4}n^2 + \frac{1}{4} - n \ln n \leq 1 \ln 1 + 2 \ln 2 + \cdots + (n-1) \ln(n-1) \leq \frac{1}{2}n^2 \ln n - \frac{1}{4}n^2 + \frac{1}{4}$.
 - $1 \cdot e^{-1^2} + 2 \cdot e^{-2^2} + \cdots + n \cdot e^{-n^2} \leq 3/(2e)$.
- Find exact closed-form solutions to the following recurrences.
 - $g(n) = g(n-1) + 4g(n-2)$, $g(0) = 0$, $g(1) = 1$.
 - $T(n) = 2T(n-1) + n^2$, $T(0) = 0$.
 - $F(n) = 3F(n/3) + n/3$, $F(1) = 1$, where n is a power of 3.
- Sort the following functions in increasing order of asymptotic growth. (For example, if you are given the functions n^2 , n , and 2^n , the sorted list would be $n, n^2, 2^n$.) Show that for every pair of consecutive functions f, g in your list, f is $o(g)$.
 - $e^n, n^n, (\log n)^2, 2^{2^n}, n \log n, n, 2^{n^2}$.
 - The *tower function* $T(n)$ is given by the recursive formula $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Where do the functions $T(n)$ and $T(\log n)$ (assuming n is a power of two) fit into the list in part (a)?

5. Bottle T has 4 litres of tea and bottle C has 4 litres of coffee. In each step you pour 1 litre of liquid from bottle T into bottle C, stir, pour back 1 litre of liquid from bottle C into bottle T, and stir again. (This is a recipe for **Yuanyang**.) Let $f(n)$ be the amount of coffee in bottle T after n steps.

(a) Calculate the values $f(0)$, $f(1)$, and $f(2)$.

(b) Write a recurrence for $f(n)$.

(c) Solve the recurrence from part (b).

(d) What is limiting value of $f(n)$ as n approaches infinity? How many steps do you need to perform to approach this value within 0.01 litres?

6. You want to move the Towers of Hanoi, but now you have four poles instead of three. The rules are the same: n disks are initially stacked up from largest at the bottom to smallest on top on the leftmost pole. The objective is to move them to the rightmost pole one by one so that at no point does a larger disk cover a smaller one.

Consider the following strategy: If $n \leq 10$, ignore one of the poles and apply the solution from class for three poles. If $n > 10$, recursively move the top $n - 10$ disks to the second pole, stack up the bottom 10 disks onto the last pole using the other three poles only, and then recursively move the $n - 10$ remaining disks from the second pole to the last pole.

Let $T(n)$ be the number of steps that it takes to move the whole stack of n disks.

(a) Write a recurrence for $T(n)$. Explain why your recurrence is correct.

(b) Show that the recurrence from part (a) satisfies $T(n) = O(2^{n/10})$.

(c) (**Extra credit**) Can you come up with a different strategy in which $2^{O(\sqrt{n})}$ moves are sufficient?

ESTR 2004 mini-project You have to tile a 30×1 hallway with unlimited supply of 1×1 red tiles, 2×1 red tiles, and 2×1 blue tiles. In how many ways can you tile:

(a) A 30×1 hallway for which you are allowed to use all types of tiles.

(b) A 30×1 hallway for which you are allowed to use all types of tiles, but no two red tiles are consecutive.

(c) A 30×3 hallway for which you are allowed to use all types of tiles.

(d) (**Extra credit**) A 30×3 hallway for which you are allowed to use all types of tiles, but no two red tiles are adjacent (share a side).