This homework is optional. If you turn it in for grading, you can use it to replace your lowest grade from the other five homeworks provided that you followed the homework guidelines throughout the semester.

Questions 1 to 6 are worth 10 points each. Please turn in solutions to *four* questions of your choice (both for ENGG 2440A students and for ESTR 2004 students).

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without credit will be considered plagiarism and may result in failing the whole course.

Questions

1. How many 5-card poker hands are there in which:
   (a) At least two suits are present in the hand?
   (b) All five cards have different face values?
   (c) There are as many hearts as there are spades?

2. How many seating arrangements are there with six boys and six girls if
   (a) The table is triangular with 4 seats on each side. Two arrangements are identical if they differ by a turn of the table that keeps the chairs in place.
   (b) The table is hexagonal with 2 seats on each side, out of which one must be occupied by a boy and the other one by a girl.
   (c) The table is triangular (4 seats on each side) and at least one girl must sit on each side.

3. Use the pigeonhole principle to prove the following propositions.
   (a) 250 courses have their final exams scheduled over a period of 20 days. There are 3 possible time slots in which an exam can be held on a given day. At most two exams can take place in the same venue at the same time. Show that at least 3 venues need to be booked.
   (b) Among every 1001 points in a three-dimensional cube of side length 1 there is a pair within distance 0.18.
   (c) Every set S of 20 integers between 0 and 188 contains four distinct elements x, y, z, w such that 189 divides $x - y + z - w$. (**Hint:** Consider their pairwise sums modulo 189.)

4. Use the inclusion-exclusion principle to answer the following questions:
   (a) Is it possible to have 45 students and some number of clubs so that each club has at most 20 students, every two clubs have at least 6 students in common, but no three clubs have a student in common?
   (b) In how many ways can you partition the integers between 1 and 100 into 3 nonempty sets? (**Hint:** Calculate the size of the complement.)

5. Let X be the set of bit sequences of length $n - k$ with exactly k zeros and Y be the set of n bit sequences that have k zeros no two of which are in consecutive positions and that always end in a 1. For example, if $n = 7$ and $k = 3$ then $(0, 1, 1, 0, 1, 0, 1) \in Y$ but $(0, 1, 1, 1, 0, 0, 1), (0, 1, 0, 1, 0, 1, 0) \notin Y$.

   Let $f \colon X \to Y$ be the function that does the following operation to its input bits from left to right: (1) If the input bit is a 1, output a 1. (2) If the input bit is a 0 output the pair 0, 1. For example, $f(0, 1, 0, 0) = (0, 1, 1, 0, 1, 0, 1)$.

   (a) Show that f is injective.

   (b) Show that f is surjective.

   (c) How many bit sequences are there with z zeros, o ones in which there is no pair of consecutive zeros?

   (d) (**Extra credit**) How many circular arrangements of z zeros and o ones are there in which there is no pair of consecutive zeros?

6. In this question you will prove a lower bound on the number of edges of any switching network.

   (a) Let G be a directed acyclic graph with n vertices whose respective out-degrees are d_1, d_2, up to d_n. Show that the number of possible collections of vertex-disjoint paths in G is at most $(d_1 + 1) \cdot (d_2 + 1) \cdots (d_n + 1)$.

   (b) Show that if G is a switching network for k packets then $k! \leq (d_1+1) \cdot (d_2+1) \cdots (d_n+1)$.

   (c) The *arithmetic-geometric mean inequality* states that for all real numbers $x_1, \ldots, x_n \geq 0$ and all integers $n \geq 0$,

   $$x_1 \cdot x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^n$$

   Prove this inequality using induction in the special case when n is of the form 2^t.
   (Extra credit) Can you deduce the general case from this special case?

   (d) Use parts (b) and (c) derive an lower bound for the number of edges in a switching network for 100, 1,000, and 10,000 packets.

   (e) Show that any switching network for k packets must have $\Omega(k \log k)$ directed edges.