Questions 1 to 6 are worth 10 points each. Please turn in solutions to *four* questions of your choice. Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and give credit to your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without reference will be considered plagiarism and may result in failing the whole course.

Questions

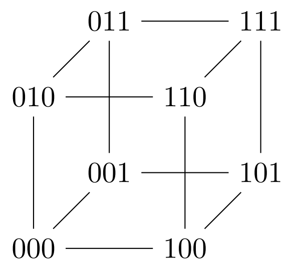
1. In which of the following Die Hard scenarios does Bruce survive? Justify your answer.
   (a) Target 12ℓ, jug capacities 182ℓ and 217ℓ.
   (b) Target 6ℓ, jug capacities 16ℓ, 28ℓ, and 36ℓ.
   (c) Target $\frac{1}{2}\ell$, jug capacities $6\frac{1}{4}\ell$ and $11\frac{1}{4}\ell$.
   (d) **(Extra credit)** Target 1ℓ, jug capacities 1731ℓ and 1255ℓ and the task must be accomplished within 140 steps (transitions).

2. Do the following graphs exist? If yes, give an example. If no, prove that it doesn't.
   (a) A graph with 100 vertices of degree 3 and 3 vertices of degree 99.
   (b) A graph with 100 vertices of degree 2 and 2 vertices of degree 99.
   (c) A graph with 99 vertices of degree 3 and 9 vertices of degree 43.

3. Solve the following (systems of) equations in modular arithmetic. Justify your steps.
   (a) $2440x \equiv 1037 \pmod{7333}$. (7333 is a prime number.)
   (b) $3x + 8y \equiv 19 \pmod{23}$
   $10x + 13y \equiv 22 \pmod{23}$.
   (c) $x + y + z \equiv 0 \pmod{2}$
   $x + y + w \equiv 0 \pmod{2}$
   $x + z + w \equiv 0 \pmod{2}$
   $y + z + w \equiv 1 \pmod{2}$.

4. Which of the following graphs $G = (V, E)$ has a perfect matching? If a graph has a perfect matching, describe it. If not, prove that a perfect matching does not exist.
   (a) The vertices are the squares of an 8 by 8 chessboard. The edges are pairs of vertices that are adjacent along a row or column.
   (b) Same as part (a), but now the bottom left and upper right corners of the chessboard are removed.
   (c) The vertices are all integers between 51 and 120 inclusively. The edges are those $\{x, y\}$ for which $3x + 5y > 600$.
   (d) The vertices are all integers between -67 and 67 inclusively except for 0. The edges are those $\{x, y\}$ for which $-1156 \leq x \cdot y < 0$.

5. The *hypercube* H_n of dimension n is the following graph on 2^n vertices: The vertices of H_n are all $\{0, 1\}$ strings of length n. There is an edge for any two vertices that differ in exactly one bit position. Here is a diagram of H_3:



   (a) Show that for every $n \geq 1$, H_n is a bipartite graph.

   (b) Show that for every $n \geq 1$, H_n has a perfect matching.

   (c) Now assume that n is odd and let G_n be the graph obtained by removing all vertices from H_n except those that have exactly $(n-1)/2$ zeroes or $(n-1)/2$ ones. Give perfect matchings for the graphs G_3 and G_5.

   (d) Prove that for every odd $n \geq 1$, G_n has a perfect matching. (**Hint:** Use Hall's theorem.)

6. Consider the following procedure:

   PROCEDURE P:
   INPUT: k positive integers $a_1, a_2, \ldots, a_k$.
   1 Assign the value a_i to variable x_i for all i between 1 and k.
   2 While there exists a pair $x_i < x_j$ such that x_i does not divide x_j:
   3   Replace x_i and x_j by $\gcd(x_i, x_j)$ and $x_i \cdot x_j / \gcd(x_i, x_j)$, respectively.
   4 Output the largest number among $x_1, x_2, \ldots, x_k$.

   (a) What are the possible outputs of $P(6, 10, 15)$? (The output might depend on the choice of x_i and x_j in step 2.)

   (b) Show that the output of P must divide the product $a_1 a_2 \cdots a_k$ by formulating and proving a suitable invariant.

   (c) Show that for positive a, b, c, if $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$ then $\gcd(ab, c) = 1$. (**Hint:** Use the connection between gcds and integer combinations.)

   (d) Use part (c) to show that if $\gcd(a_i, a_j) = 1$ for all $i \neq j$ then the output of P must equal $a_1 a_2 \cdots a_k$.

   (e) Show that the output of P cannot equal $a_1 a_2 \cdots a_k$ unless $\gcd(a_i, a_j) = 1$ for all $i \neq j$.

   (f) **(Extra credit)** Show that P always terminates.