ENGG 2440A: Discrete Mathematics for Engineers | Homework 6
The Chinese University of Hong Kong, Fall 2017 | due Thursday 7 December

Questions 1 to 6 are worth 10 points each. Please turn in solutions to *four* questions of your choice. Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and give credit to your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without reference will be considered plagiarism and may result in failing the whole course.

Questions

1. There are n boys and n girls. In how many ways can you
   (a) divide them into n $\{\text{boy}, \text{girl}\}$ pairs?
   (b) divide them into n unordered pairs without constraints on gender?
   (c) divide them into n unordered pairs of the same gender? Assume n is even.

2. This question concerns 5-card poker hands. Assume that all hands are equally likely. What is the probability that the hand has
   (a) an odd number of hearts?
   (b) exactly two cards with the same face value?
   (c) at least one heart and at least one spade, and the face value of the largest heart is the largest value of the largest spade?
   (d) Your opponent has been dealt $\{9\clubsuit, 9\diamondsuit, 9\heartsuit, 9\spadesuit, 5\spadesuit\}$. Your hand is chosen from the remaining cards. What is the probability you beat her? You need a four-of-a-kind with a face value larger than 9 or a straight flush (five consecutive cards of the same suit).

3. What is the probability of the seating arrangements happen with four boys and six girls in the following table settings? Assume all arrangements are equally likely.
   (a) They sit in a line and no two boys sit together.
   (b) They sit in a circle and no two boys sit together. (Two arrangements are identical if they differ by a turn of the circle.)
   (c) They sit in a line and no three girls sit together.

4. Use the inclusion-exclusion principle to answer the following questions:
   (a) Is it possible to have 19 students and some number of clubs so that each club has at most 10 students, every two clubs have at least 4 students in common, but no three clubs have a student in common?
   (b) In how many ways can you assign 20 *different* balls into 5 different bins so that there are no empty bins? (**Hint:** Calculate the size of the complement.)

5. Let Y be the set of arrangements of n stars and k bars in which no two bars are consecutive. For example when $n = 6$ and $k = 3$ then $\star\star\,|\,\star\,|\,\star\star\star\,|$ is in Y, and $\star\star\,|\,|\,\star\,|\,\star\star\star$ is not in Y.

   (a) List all the arrangements in Y when $n = 6$ and $k = 3$. How many are there?

   (b) In general, what is the size of Y? Justify your answer.

   (c) Let X be the set of all arrangements of $n - k + 1$ stars and k bars. For $x \in X$, let $f(x)$ be the sequence obtained by replacing every $|$ in x by $|\,\star$. For example, $f(\star\,|\,|) = \star\,|\star|\,\star$. Prove that f is a bijection from X to $Y \times \{\star\}$.

6. In this question you will investigate the best possible advantage for the second player in the game of intransitive dice. Given a set D of three dice, Alice chooses a die, Bob chooses on of the remaining two dice, then they each toss their die and the higher number wins. Let $p(D)$ be the probability that Bob wins, assuming they both choose their die optimally.

   (a) Calculate the value of $p(D)$ for the following three dice:

   Die A: $3, 3, 3, 3, 3, 6$ Die B: $2, 2, 2, 5, 5, 5$ Die C: $1, 4, 4, 4, 4, 4$.

   (b) Let $D = \{A, B, C\}$ be any set of three n-sided dice. Consider the experiment in which all three dice are tossed. Describe the sample space and the probabilities of all possible outcomes.

   (c) If the face values of the outcome are (a, b, c), let E_{AB} be the event $a > b$, E_{ABC} be the event $a > b > c$, and so on. Assume the dice have disjoint sets of face values. Show that

$$\Pr[E_{AB}] + \Pr[E_{BC}] = 1 + \Pr[E_{ABC}] - \Pr[E_{CBA}].$$

   (d) Use part (c) to show that for every D, $p(D) \leq 2/3$.

   (e) **(Extra credit)** Can you find a set of dice (no restriction on the number of faces) for which $p(D)$ is larger than the value you calculated in part (a)?