1. Underline and explain the mistake in the following "proof."

   Theorem. Every graph has a vertex of even degree.

   Proof. By induction on the number of vertices n. When $n = 1$ the graph has a vertex of degree zero, which is even. Now assume it is true for graphs with n vertices. Let G be a graph with $n + 1$ vertices. Remove any vertex from G. By inductive hypothesis the remaining graph G' has a vertex v of even degree. Since v is also a vertex of G, G has a vertex of even degree. □

   Solution: If v has even degree in G' we cannot conclude that v has even degree in G. The degrees of v in G and G' may be of different parity. For example, if G has two vertices and one edge then v has degree 1 in G but it has degree 0 in G'.

2. Prove that for every integer n there exists an integer k such that $|n^2 - 5k| \leq 1$.

   Solution: First we check that for all n, n^2 mod 5 equals 0, 1 or 4:

n mod 5	0	1	2	3	4
n^2 mod 5	0	1	4	4	1

   Since $4 \equiv -1 \pmod 5$ it follows that for every n, n^2 is congruent to 0, 1, or -1 modulo 5. Therefore n^2 is of the form $5k$ or $5k - 1$ or $5k + 1$ for some integer k. In all cases $|n^2 - 5k| \leq 1$.

3. Alice has infinitely many \$6, \$10, and \$15 stamps. Can she make all integer postages above \$30?

   Solution: Alice can make all integer postages from \$30 to \$35 as follows:

$$\begin{aligned}
\$30 &= 5 \times \$6 \\
\$31 &= \$6 + \$10 + \$15 \\
\$32 &= 2 \times \$6 + 2 \times \$10 \\
\$33 &= 3 \times \$6 + \$15 \\
\$34 &= 4 \times \$6 + \$10 \\
\$35 &= 2 \times \$10 + \$15
\end{aligned}$$

   Now we show that she can make any amount n above 30 by strong induction on n. We already covered the cases $30 \leq n \leq 35$. Now assume that $n > 35$ and she can make all amounts between \$30 and \$$n$. Then $n - 6 \geq 30$ and by inductive assumption she can make $n - 6$ dollars. By adding one \$6 stamp she obtains n dollars.

4. Bob has 32 $b$lue, 33 $r$ed, and 34 $g$reen balls. At every turn he takes out two balls and replaces them with two different balls by the rule below. Can he obtain 99 balls all of the same color?

   replacement rule: $bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb$

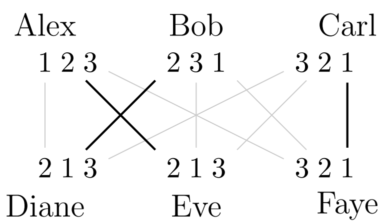
Solution: We can represent this process by a state machine with states (B, R, G) indicating the number of balls of each color, start state $(32, 33, 34)$, and transitions from (B, R, G) to the states $(B-1, R-1, G+2)$, $(B+2, R-1, G-1)$, $(B-1, R-1, G+2)$, $(B+1, R-2, G+1)$, $(B-2, R+1, G+1)$, $(B+1, R+1, G-2)$ as long as all numbers remain non-negative. The predicate $R - B \equiv 1 \pmod 3$ is an invariant: It holds in the start state and it is preserved by all transitions as $R - B$ can only change by -3, 0, or 3. If all 99 balls are of the same color then $R - B \equiv 0 \pmod 3$, so such a state cannot be reached.

5. A summer camp has children from Hong Kong, Mumbai, and Tokyo. The table entry in row i and column j gives the average number of friends from city j that children from city i report to have. Prove that not all reports are accurate.

	H	M	T
H	2	3	3
M	3	5	1
T	4	2	3

Solution: Suppose for contradiction that all reports are accurate and let H, M, and T be the sets of Hong Kong, Mumbai, and Tokyo children in the camp. If we look at the bipartite graph of friendships between sets H and T, by the handshaking lemma from Lecture 5 we get that $3|H| = 4|T|$ (both are equal to the total number of edges between H and T). By the same reasoning applied to the other two pairs we get that $3|M| = 3|H|$ and $2|T| = |M|$. Multiplying both sides of these equations we obtain that $3|H| \cdot 3|M| \cdot 2|T| = 4|T| \cdot 3|H| \cdot |M|$, from where $18 = 12$. Contradiction.

6. Find a stable matching for these preferences and show that there is no other stable matching.



Solution: Consider the marked matching {Alex, Eve}, {Bob, Diane}, {Carl, Faye}. We show that no other matching is stable. As a stable matching always exists, this one must be stable.

In any stable matching, Carl must be matched to Faye because they are each other's first choice (so they would be a rogue couple if not matched). For the rest, the matching {Alex, Diane}, {Bob, Eve} can be ruled out because Bob and Diane would be a rogue couple. This leaves the above matching as the only stable possibility.

Alternative solution: If we run the Gale-Shapley algorithm, on day 1 Alex proposes to Diane and Bob and Carl propose to Faye. Faye picks Carl, so on day 2 both Alex and Bob propose to Diane. Diane picks Bob, so the final matching is {Alex, Eve}, {Bob, Diane}, {Carl, Faye}. We proved in Lecture 5 that this is stable.

Let us now run the Gale-Shapley algorithm again, but with the girls doing the proposing this time around. On day 1 Diane and Eve propose to Bob and Faye proposes to Carl. Carl picks Faye and Bob picks Diane over Eve. On day 2 Eve proposes to Alex resulting in the same final stable matching.

By Theorem 6 in lecture 5, the first matching is the best possible for the boys (every boy gets his best possible choice among all stable matchings), while the second one is the worst possible for the boys (every boy gets his worst possible choice). Since they are the same there can be only one stable matching.