Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions, give credit to your collaborators on your solution sheet, and follow the faculty guidelines and university policies on academic honesty regarding the use of external references.

Questions

1. In which of the following Die Hard scenarios does Bruce survive? Justify your answer.
   (a) Target 14ℓ, jug capacities 35ℓ and 63ℓ.
   (b) Target 7ℓ, jug capacities 12ℓ, 18ℓ, and 30ℓ.
   (c) Target $\frac{1}{2}\ell$, jug capacities $8\frac{1}{4}\ell$ and $14\frac{1}{4}\ell$.

2. Do the following graphs exist? If yes, give an example. If no, prove that it doesn't.
   (a) A graph with 14 vertices of degree 3 and 3 vertices of degree 7.
   (b) A graph with 10 vertices of degree 2 and 2 vertices of degree 9.
   (c) A graph with 10 vertices of degree 4 and 9 vertices of degree 14.

3. A summer camp has children from Hong Kong, Shanghai, and Tokyo. The table entry in row i and column j gives the average number of friends from city j that children from city i report to have. Find the missing entry. Justify your answer.

	H	S	T
H	2	?	4
S	3	5	1
T	6	2	3

4. Calculate the following numbers in modular arithmetic. Justify your answers.
   (a) What is $9^{-1} \pmod{23}$?
   (b) Suppose $5x + 7y \equiv 17 \pmod{19}$ and $4x + 11y \equiv 13 \pmod{19}$. What are x and y?
   (c) What is $1^1 + 2^2 + \cdots + 99^{99} \pmod{3}$?

5. Which of the following graphs $G = (V, E)$ has a perfect matching? If a graph has a perfect matching, describe it. If not, prove that a perfect matching does not exist.
   (a) The vertices are the squares of a 9 by 9 chessboard with the top left corner removed. The edges are pairs of vertices that are adjacent along a row or column.
   (b) Same as part (a), but now the chessboard is 10 by 10, and both the top left and bottom right corners are removed.
   (c) The vertices are all integers between 53 and 97 inclusively. The edges are those $\{x, y\}$ for which $3x + 4y > 500$.
   (d) The vertices are all integers between -23 and 23 inclusively except for 0. The edges are those $\{x, y\}$ for which $-144 \leq x \cdot y < 0$.

6. Consider the following procedure:

PROCEDURE P:
INPUT: k positive integers $a_1, a_2, \ldots, a_k$.

1 Assign the value a_i to variable x_i for all i between 1 and k.
2 While there exists a pair $x_i < x_j$ such that x_i does not divide x_j:
3   Replace x_i and x_j by $\gcd(x_i, x_j)$ and $x_i \cdot x_j / \gcd(x_i, x_j)$, respectively.
4 Output the largest number among $x_1, x_2, \ldots, x_k$.

(a) What are the possible outputs of $P(6, 10, 15)$? (The output might depend on the choice of x_i and x_j in step 2.)

(b) Show that the output of P must divide the product $a_1 a_2 \cdots a_k$ by formulating and proving a suitable invariant.

(c) Show that for positive a, b, c, if $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$ then $\gcd(ab, c) = 1$. (**Hint:** Use the connection between gcds and integer combinations.)

(d) Use part (c) to show that if $\gcd(a_i, a_j) = 1$ for all $i \neq j$ then the output of P must equal $a_1 a_2 \cdots a_k$.

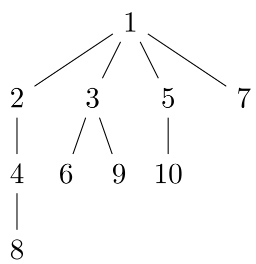
(e) Show that the output of P cannot equal $a_1 a_2 \cdots a_k$ unless $\gcd(a_i, a_j) = 1$ for all $i \neq j$.

(f) **(Extra credit)** Show that P always terminates.

ESTR 2004 mini-project In this project you will seek the smallest vertex cover that you can find for the following graphs. You may use any of the algorithms described in Lecture 5, an algorithm that you devise, mathematical reasoning, or any mix of methods that you find suitable.

(a) The vertices are the integers from 0 to 1023, and $\{u, v\}$ is an edge if $\log_2 |u - v|$ is an integer.

(b) The vertices are the integers from 1 to 100, and $\{u, v\}$ is an edge if and only if u is that largest number less than v that divides v. This is the part induced by vertices 1 to 10:



(c) The vertices are four-symbol strings $x_1 x_2 x_3 x_4$, where each x_i is 0, 1, 2, 3, or 4. The pair $\{x_1 x_2 x_3 x_4, y_1 y_2 y_3 y_4\}$ is an edge if and only if for every i, $x_i - y_i$ is 0, 1, or 4 modulo 5.

Describe your solution clearly. Explain how it compares to the best possible. If you used a computer program to calculate the vertex cover explain how it works.