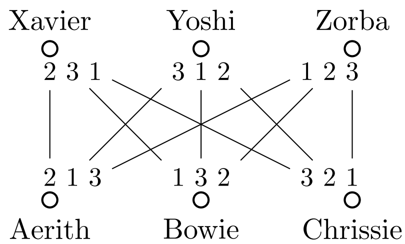
ENGG 2440A / ESTR 2004: Discrete Mathematics for Engineers — Homework 4
The Chinese University of Hong Kong, Fall 2018 — due Thursday 15 November

Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions, give credit to your collaborators on your solution sheet, and follow the faculty guidelines and university policies on academic honesty regarding the use of external references.

Questions

1. Consider 3 boys (Xavier, Yoshi, and Zorba) and 3 girls (Aerith, Bowie, and Chrissie) with the following preference lists:



   (a) Describe the execution of Gale-Shapley algorithm on these preference lists with the boys proposing. Show the stable matching produced by the algorithm.

   (b) Repeat part (a), but now with the girls proposing.

   (c) Is there a stable matching different both from the one in part (a) and the one in part (b)? Justify your answer.

2. Find exact closed form expressions for the following sums. Explain how you discovered the expression and prove that it is correct.

   (a) $1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n$

   (b) $1^3 + 2^3 + \cdots + n^3$

   (c) $n \cdot 1^2 + (n-1) \cdot 2^2 + \cdots + 1 \cdot n^2$

   (d) **(Extra credit)** $(0^2 + \cdots + n^2) + (1^2 + \cdots + (n+1)^2) + \cdots + (n^2 + \cdots + (2n)^2)$

3. Show the following inequalities.

   (a) $1.222 \le 1 + 3^{-2} + 5^{-2} + 7^{-2} + \cdots \le 1.252$

   (b) $2\sqrt{n} - 2 + \frac{1}{\sqrt{n}} \le 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \le 2\sqrt{n} - 1$

   (c) $1 - 2(n+1)^{-2} \le \frac{4}{1\cdot 2\cdot 3} + \cdots + \frac{4}{n(n+1)(n+2)} \le 1 - 2(n+2)^{-2}$
   (**Hint:** $\frac{(x+2)-x}{x(x+1)(x+2)} = \frac{1}{x(x+1)} - \frac{1}{(x+1)(x+2)}$)

4. Are the following propositions about digraphs true or false? Justify your answer by giving a proof or a relevant example.

   (a) The sum of in-degrees of a digraph equals the sum of out-degrees. The in-degree and out-degree of a vertex is its number of incoming and outgoing edges, respectively.

   (b) There exists a balanced DAG with at least one edge. A digraph is *balanced* if for every vertex v the in-degree of v equals the out-degree of v.

   (c) For every $n \geq 1$, there is a digraph with vertices $1, \ldots, n$ such that the in-degree of vertex v is $n - v$ and its out-degree is $v - 1$.

5. A k-regular forest is a forest all of whose vertices have degree 1 or degree k.

   (a) Prove that if G is a 3-regular forest then $n = 2(\ell - c)$ where n, c, ℓ is the number of vertices, connected components, and leaves in G, respectively. (**Hint:** Induction on n.)

   (b) Prove that if G is a k-regular forest, then $(k-2)n = (k-1)\ell - 2c$, and n or ℓ must be even, for every $k \geq 3$.

   (c) Prove that a k-regular *tree* (i.e., $c = 1$) G exists if and only if $n - 2$ is a multiple of $k - 1$. (**Hint:** Apply induction on the number of degree-k vertices)

6. Let G be the digraph whose vertices are the 125 3-digit numbers with digits 0, 1, 2, 3, 4 (leading zeros are allowed), and (u, v) is an edge if $v - u$ equals 1, 10, or 100.

   (a) Show that G is acyclic.

   (b) What is the duration of the shortest parallel schedule for G? Justify your answer.

   (c) Show that G has an antichain of size 19.

   (d) Show that the vertices of G can be partitioned into 19 (vertex-disjoint) paths. Conclude that G does not have an antichain of size 20.

ESTR 2004 mini-project Consider the following types of graphs on n vertices:

- A type A graph is the union of two edge-disjoint cycles, each of length n. (The two cycles cannot share any edges.)
- A type B graph is the union of two distinct cycles, each of length n. (The two cycles may share some edges but they are not identical.)

Let $s_{\mathrm{A}}(n)$ and $s_{\mathrm{B}}(n)$ be the maximum possible size of a shortest cycle among all type A graphs and type B graphs on n vertices, respectively.

1. What is $s_{\mathrm{A}}(8)$ and $s_{\mathrm{B}}(8)$?
2. Calculate the exact values of $s_{\mathrm{A}}(n)$ and $s_{\mathrm{B}}(n)$ for as many values of n as you can.
3. Obtain the best upper and lower bounds you can for $s_{\mathrm{A}}(9,900)$ and $s_{\mathrm{B}}(9,900)$.