1. Write the proposition "Every class except possibly `ENGG8888` meets at least twice a week" using logical connectives, quantifiers, and the predicate $M(c, d)$ for "class c meets on day d of the week".

   Solution: $\forall c : c \neq \texttt{ENGG8888} \longrightarrow \exists d, d' : M(c, d)$ AND $M(c, d')$ AND $d \neq d'$.

2. Prove that the number $(\sqrt{2} - 1)/(\sqrt{8} - 1)$ is irrational.

   Solution: Assume for contradiction that $(\sqrt{2} - 1)/(\sqrt{8} - 1) = r$ for some rational number r. Multiplying both sides by $\sqrt{8} - 1$ and simplifying, we obtain that $(1 - 2r)\sqrt{2} = 1 - r$. If r equals $1/2$ we get that $0 = 1/2$ which is impossible. Otherwise, we can divide both sides by $1 - 2r$ to obtain $\sqrt{2} = (1 - r)/(1 - 2r)$, which is a rational number, contradicting the fact that $\sqrt{2}$ is irrational.

3. What is the multiplicative inverse of 100 modulo 1009? Show your work.

   Solution: We look for integers s and t such that $100s + 1009t = 1$. Applying division with remainder as in Euclid's algorithm, we can write

$$\begin{aligned} 1009 &= 100 \cdot 10 + 9 \\ 100 &= 9 \cdot 11 + 1, \end{aligned}$$

   from where

$$1009 \cdot 11 = 100 \cdot 110 + 9 \cdot 11 = 100 \cdot 110 + 100 - 1 = 100 \cdot 111 + 1,$$

   so we can set $s = 111$ and $t = -11$. Therefore $100 \cdot 111 \equiv 1 \mod 1009$ and 111 is the desired multiplicative inverse.

4. Prove that for all positive integers n, $(3n)! < 7^n \cdot n! \cdot (2n)!$.

   Solution: We apply induction on n. When $n = 1$, $(3n)! = 6$ and $7^n \cdot n! \cdot (2n)! = 14$ so the claim is true. Suppose it is true for n. Then

$$\begin{aligned} (3(n+1))! &= (3n)! \cdot (3n+1)(3n+2)(3n+3) \\ &\leq 7^n \cdot n! \cdot (2n)! \cdot (3n+1)(3n+2)(3n+3) && \text{by inductive assumption} \\ &= 7^n \cdot n! \cdot (2n)! \cdot 27(n+\tfrac{1}{3})(n+\tfrac{2}{3})(n+1) \\ &\leq 7^n \cdot n! \cdot (2n)! \cdot 28(n+\tfrac{1}{2})(n+1)(n+1) \\ &= 7^{n+1} \cdot n! \cdot (2n)! \cdot (2n+1)(2n+2)(n+1) \\ &= 7^{n+1} \cdot (n+1)! \cdot (2(n+1))!, \end{aligned}$$

   so it is also true for $n + 1$, and it is therefore true for all n.

5. Start with the number 0123456789. A *pancake flip* replaces some four consecutive digits *abcd* by *dcba*, e.g., $3096\underline{1258}47 \to 3096\underline{1485}27$. Can you obtain 9876543210 by a sequence of pancake flips?

   Solution: No. First we prove that "the number of inversions is even" is an invariant. As usual an inversion is a pair of digits whose order is flipped (i.e., the larger one is to the left). In the initial state there are no inversion so the invariant holds. Now suppose the invariant holds before a pancake flip $abcd \to dcba$. Before the pancake flip, the number of inversions is $i+n$, where i counts the inversions among the pairs $\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}$, and n counts all the others. After the flip there are $(6-i)+n$ inversions. Since $i+n \equiv (6-i)+n \mod 2$ the evenness of the inversion count is preserved by the transition.

   The state 9876543210 has $9+8+7+6+5+4+3+2+1 = 9\cdot 10/2 = 45$ inversions, which is an odd number, so it cannot be reached.

6. Prove that in every stable matching of 9 boys and 9 girls (with arbitrary preference lists) at least one person is matched with one of their top 5 choices.

   Solution: First we prove

   Lemma X. There exist a boy and a girl that are among each other's top 5 choices.

   Proof: For contradiction suppose there is no such boy-girl pair. Then if a girl is among a boy's top 5 choices, the boy must be one of her last 4 choices. Let G be the bipartite graph whose vertices are the 9 boys and the 9 girls and whose edges are the pairs $\{b,g\}$ where g is among b's top 5 choices. Each boy has degree 5 in G, while each girl has degree at most 4 (since her neighbors must be among her last 4 choices). This is in contradiction to the handshaking lemma, by which the average degrees of the boys and the girls are the same. □

   Now suppose Ξ is a matching in which noone is matched with one of their top 5 choices. By Lemma X there is a boy and a girl that would rather be with each other than with their partner in Ξ, so Ξ cannot be stable.

   Alternative proof of Lemma X: Let B be the bipartite graph of boys and girls where there is an edge between every boy and his last 4 choices. Let G be the graph with the same vertices, but whose edges are between every girl and her last 4 choices. Since each boy has degree 4 in B and each girl has degree 4 in G, together B and G have $2\cdot 4\cdot 9 = 72$ edges. As there are 81 boy-girl pairs, at least one of them is not an edge in either graph, so this boy and girl are among each other's top 5 choices. □