Practice questions

1. A point is chosen uniformly at random inside a circle with radius 1. Let X be the distance from the point to the centre of the circle. What is the (a) CDF (b) PDF (c) expected value and (d) variance of X? *[Adapted from textbook problem 3.2.7]*

2. Bob's arrival time at a meeting with Alice is X hours past noon, where X is a random variable with PDF
$$f(x) = \begin{cases} cx, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Find the value of the constant c.
   (b) What is the probability that Bob arrives by 12.30?
   (c) What is the expected hour of Bob's arrival?
   (d) Given that Bob hasn't arrived by 12.30, what is the probability that he arrives by 12.45?
   (e) Given that Bob hasn't arrived by 12.30, what is the expected hour of Bob's arrival?

3. Alice arrives at her bus stop at noon. Buses arrive at a rate of 3 per hour in the next hour and 1 per hour after that.
   (a) Divide each hour into n equal intervals and let E_i be the event "a bus arrives in the i-th interval past noon." What is $\mathrm{P}(E_i)$? (Assume n is sufficiently large so that the probability of two or more buses arriving in interval i is negligible.)
   (b) Let I_n be the index of the interval in which the first bus arrives. Assuming the events E_i are independent, what is the CDF of I_n?
   (c) Let T be a random variable whose CDF is $F_T(t) = \lim_{n\to\infty} \mathrm{P}(I_n/n \leq t)$. Calculate the CDF of T. What does T represent?
   (d) Calculate the PDF and the expected value of T.

4. To send a message $m \in \{-1, 1\}$ to Bob, Alice emits a signal mx of "strength" $x > 0$. Owing to noise Bob receives a Normal$(mx, 1)$ random variable Y and decodes it to the sign of Y ($+1$ if Y is positive, -1 if negative). The cost of operating this scheme is x cents if the decoding is correct and $x + 10$ cents if it isn't. How should Alice pick x to minimize the expected cost?

Additional ESTR 2018 questions

5. Let $p(\sigma)$ be the probability of the event "$\lfloor X \rfloor$ is even", where X is a Normal$(0, \sigma)$ random variable and $\lfloor x \rfloor$ is the largest integer that is no larger than x. What is $p_\infty = \lim_{\sigma\to\infty} p(\sigma)$? Can you plot $p(\sigma)$ in terms of σ? For which σ does $|p(\sigma) - p_\infty|$ drop and stay below 10^{-4}? (It is okay to investigate this question empirically.)

6. Suppose a random variable X with PDF f that has the following two properties: (1) $f(x) > 0$ for every real number x and (2) $f(x)/f(y) \leq 2$ whenever $|x - y| \leq 1$. What is the smallest possible variance that X can have?