1. Roll a 4-sided die twice. Let $X$ be the larger number and $Y$ be the smaller number you rolled. Find (a) the conditional PMF of $X$ given $Y$ and (b) $\mathrm{E}[X \mid Y=y]$ for $y=1,2,3,4$.
2. Cup 1 contains three $\$ 1$ coins. Cup 2 contains a $\$ 1$ coin, a $\$ 2$ coin, and a $\$ 5$ coin. Alice chooses a random cup, takes out two coins, and gives the second coin to Bob.
(a) How many dollars does Bob expect to gain?
(b) Bob sees that the first coin out is a $\$ 1$ coin. Did his expected gain increase?
3. Express $X$ and $Y$ below as $1+A+B$ for some Geometric random variables $A$ and $B$. Use this formula to find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$. Are $A$ and $B$ independent? If yes, find $\operatorname{Var}[X]$ and $\operatorname{Var}[Y]$.
(a) The first time $X$ at which both a TH and a HT have appeared in a sequence of coin flips.
(b) The first time $Y$ at which 1, 2, and 3 have all appeared in a sequence of 3 -sided die rolls.
4. You roll a six-sided die and the value is $X_{1}$. You can either cash in $X_{1}$ dollars, or choose to roll again, in which case you cash in the value $X_{2}$ of the second roll in dollars.
(a) For which values of $X_{1}$ should you roll again in order to maximize your expected utility?
(b) What is your expected utility for the strategy in part (a)?
(c) You can now roll a third time if you are unhappy with $X_{2}$. How does this change your answer in (a) and (b)? (Hint: What is your maximum expected utility given $X_{1}=x_{1}$ ?)
(d) (Extra credit) What is your maximum expected utility if you can roll up to $t$ times?
[Based on Blitzstein-Hwang exercise 4.4]

## Additional ESTR 2018 questions

5. Let $X$ be the number of times a 6 -sided die is rolled until a 6 appears. Let $A$ be the event that all rolls before the first 6 are even numbers. This question is about the value $\mathrm{E}[X \mid A]$. [due to E. Mossel]
(a) Let $Y$ be the number of times a 3 -sided die with face values $2,4,6$ is rolled until a 6 appears. What is $\mathrm{E}[Y]$ ? Do you think that $\mathrm{E}[X \mid A]$ and $\mathrm{E}[Y]$ should be the same?
(b) Calculate $\mathrm{P}((X=x) \cap A)$ for $x=1,2 \ldots$
(c) Calculate the conditional PMF $\mathrm{P}(X=x \mid A)$ and use it to obtain $\mathrm{E}[X \mid A]$.
(d) Are the answers in parts (a) and (c) the same? Why not?
