1. There are 5 red balls, 4 blue balls, and 3 green balls in a bin. You draw two balls from a bin. What is the probability that

   (a) Ball 2 is red?
   (b) Ball 2 is red given that ball 1 is red, if the balls are drawn *with* replacement?
   (c) Ball 2 is red given that ball 1 is red, if the balls are drawn *without* replacement?
   (d) Ball 2 is not blue given that ball 1 is red, if the balls are drawn *without* replacement?
   (e) Ball 2 is red given that ball 1 is not blue, if the balls are drawn *without* replacement?

   Solution: Let R_1 be the event "ball 1 is red" and B_1, G_1, R_2 be defined similarly,

   (a) $\mathrm{P}(R_2) = 5/(5+4+3) = 5/12$ as the outcomes are equally likely.
   (b) $\mathrm{P}(R_2|R_1) = \mathrm{P}(R_2) = 5/12$. Owing to the replacement the color of the first ball does not affect the color of the second ball.
   (c) $\mathrm{P}(R_2|R_1) = 4/11$ because after the first ball is drawn there are 4 red balls to choose out of 11.
   (d) $\mathrm{P}(B_2^c|R_1) = 7/11$ because after the first ball is drawn there are 7 non-blue balls to choose out of 11.
   (e) The conditioned sample space given B_1^c is partitioned by the events R_1 and G_1. By the total probability theorem,

   $$\mathrm{P}(R_2|B_1^c) = \mathrm{P}(R_2|R_1)\,\mathrm{P}(R_1|B_1^c) + \mathrm{P}(R_2|G_1)\,\mathrm{P}(G_1|B_1^c) = \frac{4}{11}\cdot\frac{5}{8} + \frac{5}{11}\cdot\frac{3}{8} = \frac{35}{88}.$$

2. Alice, Bob, and Charlie are equally likely to have been born on any three days of the year. Let E_{AB} be the event that Alice and Bob were born on the same day. Define E_{BC} and E_{CA} analogously. Which of the following statements is true:

   (a) Any two of the three events E_{AB}, E_{BC}, E_{CA} are independent.
   (b) E_{AB}, E_{BC}, and E_{CA} are independent.
   (c) $E_{AB} \cup E_{BC}$ and E_{CA} are independent.

   Solution: Our sample space will consist of all triples of possible birthdays (a, b, c) where a, b, and c are numbers between 1 and 365 (we exclude February 29 to keep things simple). We assume equally likely outcomes, so all triples occur with probability 365^{-3}.

   (a) **True.** The intersection of any two events is the event that all three were born on the same day. There are 365 such outcomes, each occurring with probability 365^{-3}, so

   $$P(E_{AB} \cap E_{BC}) = P(E_{AB} \cap E_{CA}) = P(E_{BC} \cap E_{CA}) = 365^{-2}.$$

   On the other hand, probability that any two of them were born on the same day is

   $$P(E_{AB}) = P(E_{BC}) = P(E_{CA}) = 365 \cdot \frac{1}{365^2} = 365^{-1}.$$

   Since $P(E_{AB} \cap E_{BC}) = 365^{-2} = P(E_{AB}) \cdot P(E_{BC})$, the two events E_{AB} and E_{BC} are independent, and similarly for the other two pairs.

(b) **False.** $E_{AB} \cap E_{BC} \cap E_{CA}$ is also the event that all three were born on the same day, so $\mathrm{P}(E_{AB} \cap E_{BC} \cap E_{CA}) = 365^{-2}$. On the other hand $P(E_{AB}) \cdot P(E_{BC}) \cdot P(E_{CA}) = 365^{-3}$ so the three events are not independent.

(c) **False.** The event $(E_{AB} \cup E_{BC}) \cap E_{AC}$ happens exactly when all of Alice, Bob, and Charlie have the same birthday, so $\mathrm{P}(E_{AB} \cup E_{BC}) \cap E_{AC}) = 365^{-2}$. Using the union rule,

$$\mathrm{P}(E_{AB} \cup E_{BC}) = \mathrm{P}(E_{AB}) + \mathrm{P}(E_{BC}) - \mathrm{P}(E_{AB} \cap E_{BC}) = 2 \cdot 365^{-1} - 365^{-2},$$

As $(2 \cdot 365^{-1} - 365^{-2}) \cdot 365^{-1} \neq 365^{-2}$ the events are not independent.

3. Cup 1 contains 3 blue balls and 2 red balls. Cup 2 contains 2 blue balls and 8 red balls. I choose a random cup and draw a ball from it.

(a) What is the probability that it is blue?

(b) The ball is blue. What is the probability that it came from cup 1?

(c) I draw another ball from the same cup without replacement. What is the probability that it is also blue?

Solution: Let C_i be the event "cup i was chosen", B_1 be the event "the first ball is blue", and B_2 be the event "the second ball is blue".

(a) By the total probability theorem:

$$\mathrm{P}(B_1) = \mathrm{P}(B_1|C_1) \cdot \mathrm{P}(C_1) + \mathrm{P}(B_1|C_2) \cdot \mathrm{P}(C_2) = \frac{3}{5} \cdot \frac{1}{2} + \frac{2}{10} \cdot \frac{1}{2} = \frac{2}{5}.$$

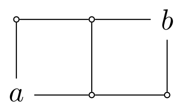
(b) By Bayes' rule

$$\mathrm{P}(C_1|B_1) = \frac{\mathrm{P}(B_1|C_1) \cdot \mathrm{P}(C_1)}{\mathrm{P}(B_1)} = \frac{3/10}{2/5} = \frac{3}{4}$$

(c) By the complement rule, $\mathrm{P}(C_2|B_1) = 1 - \mathrm{P}(C_1|B_1) = 1/4$. We use the total probability theorem again, now conditioned on B_1:

$$\mathrm{P}(B_2|B_1) = \mathrm{P}(B_2|C_1 \cap B_1) \cdot \mathrm{P}(C_1|B_1) + \mathrm{P}(B_2|C_2 \cap B_1) \cdot P(C_2|B_1) = \frac{2}{4} \cdot \frac{3}{4} + \frac{1}{9} \cdot \frac{1}{4} = \frac{29}{72}.$$

4. Computers a and b are linked through seven cables as in the picture. Each cable fails with probability 10% independently of the others. Let C be the event "there is a connection between a and b" and F be the event "the middle vertical cable fails".



(a) What is the probability of C given F?

Solution: Let T and B be the events that a connects to b via the top and bottom paths respectively. Conditioned on F, events T and B are independent so

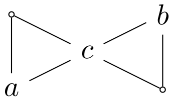
$$\mathrm{P}(C|F) = \mathrm{P}(T \cup B|F) = 1 - \mathrm{P}(T^c \cap B^c|F) = 1 - \mathrm{P}(T^c|F)\,\mathrm{P}(B^c|F).$$

Both T and B are independent of F, so by the algebra of independent events, $\mathrm{P}(T^c|F) = \mathrm{P}(T^c)$ and $\mathrm{P}(B^c|F) = \mathrm{P}(B^c)$ and we get that

$$\mathrm{P}(C|F) = 1 - \mathrm{P}(T^c)\,\mathrm{P}(B^c) = 1 - (1 - 0.9^3)^2.$$

(b) What is the probability of C given F^c?

Solution: Let c be the top middle node. Conditioned on F^c, we can contract the top and bottom single nodes and picture the network like this:



Let L and R be the events "there is a connection from a to c" and "there is a connection from c to b", respectively. They are independent so

$$\mathrm{P}(C|F^c) = P(L \cap R|F^c) = P(L|F^c)P(R|F^c) = (1 - P(L^c|F^c))(1 - P(R^c|F^c)).$$

The complement of L (given F^c) happens when both of the connections from a to c fail. Since they are independent,

$$\mathrm{P}(L^c|F^c) = 0.1 \cdot (1 - 0.9^2).$$

By symmetry, $\mathrm{P}(R^c|F^c) = 0.1 \cdot (1 - 0.9^2)$, and so

$$\mathrm{P}(C|F^c) = (1 - 0.1 \cdot (1 - 0.9^2))^2.$$

(c) What is the probability of C?

Solution: By the total probability theorem,

$$P(E) = 0.1 \cdot (1 - (1 - 0.9^3)^2) + 0.9 \cdot (1 - 0.1 \times (1 - 0.9^2))^2 \approx 0.959.$$